COSMOLOGICAL SIMULATIONS OF STRUCTURE FORMATION WITH FUZZY DARK MATTER

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Outline

What is "fuzzy dark matter"?

- F(C)DM, BECDM, ULDM, ELBDM, ψDM, quantum-wave DM, (ultra-light) axion(-like) DM (ULA, ALP)...
- New **extremely light scalar** particle ($m \approx 10^{-22}$ eV!)
  - $\rightarrow$ electron: 511 keV
- Non-thermal production mechanism (thus not ultra-hot)
- Aggregations of bosons can form a **Bose–Einstein condensate**
- Tiny mass
  - $\Rightarrow$ large de Broglie wavelength ($\lambda \sim 1/m$)
  - $\Rightarrow$ **macroscopic quantum effects** on kpc scales
  - Structures resist collapse below quantum wavelength (**Heisenberg uncertainty principle**)
- **Large scales: equivalent to cold dark matter**

Motivation for fuzzy dark matter

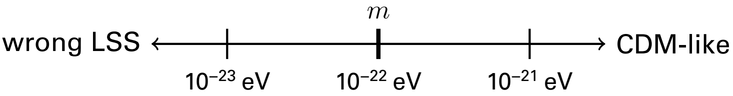
Particle physics perspective:

- Original concept – strong CP problem:
  Why doesn't QCD violate CP symmetry?
- Solved by Peccei–Quinn $\mathrm{U}(1)$ symmetry
  and (pseudo-)scalar field (*axion*!)
  Peccei and Quinn (1977)!
- Fuzzy dark matter is **not** the QCD axion, but axion-like particles are a common feature of early-universe theories

Astrophysics perspective:

- Small-scale challenges (cusp–core, missing satellites, ...)

$\rightarrow$ Ultra-light scalars: WIMP alternative, could improve this



- **No sign of (WIMP) CDM**

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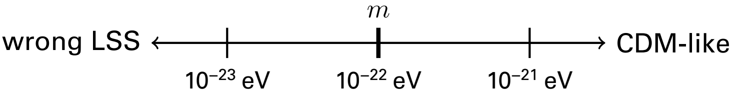

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Derivation of the fuzzy dark matter equations

Schrödinger–Poisson system

- Add a scalar field to the Einstein–Hilbert action of general relativity
$$S = \frac{1}{\hbar c^2} \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 - \frac{\lambda}{\hbar^2 c^2} \phi^4 \right)$$
  - *Superfluid DM* without self-interaction ($\lambda = 0$ or $T \to 0$)
  - *QCD axion* case: originates from periodic potential $V(\phi) \sim \Lambda^4 (1 - \cos(\phi/f_a))$ for $\phi \ll f_a$
- Rewrite
$$\phi = \sqrt{\frac{\hbar^3 c}{2m}} \frac{1}{2} \operatorname{Re}\left(\psi e^{-i c^2/\hbar\, m t}\right) = \sqrt{\frac{\hbar^3 c}{2m}} \left(\psi e^{-i c^2/\hbar\, m t} + \psi^* e^{i c^2/\hbar\, m t}\right)$$
and take non-relativistic limit with perturbed FRW metric $\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - a(t)^2 \left(1 - \frac{2\Phi}{c^2}\right) \mathrm{d}\vec{x}^2$
- Result: Schrödinger equation
$$i\hbar \left(\partial_t \psi + \frac{3}{2} H \psi \right) = -\frac{\hbar^2}{2m} \nabla^2 \psi + m \Phi \psi$$

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The fuzzy dark matter equations

Schrödinger–Poisson system

- Schrödinger equation:

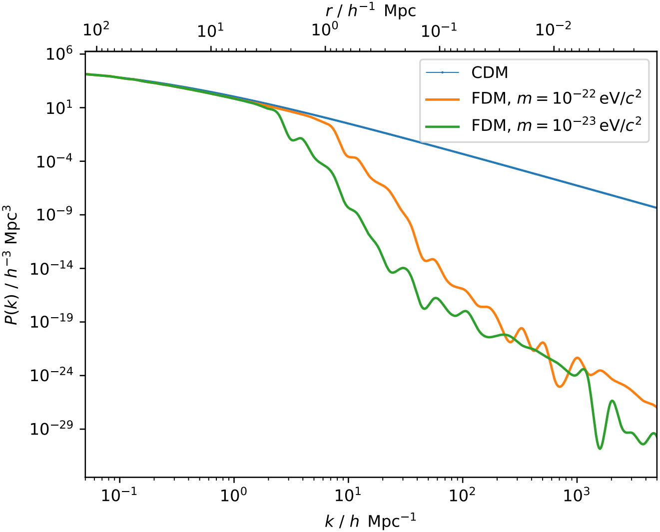
$$i\hbar\left(\partial_t\psi + \frac{3}{2}H\psi\right) = -\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi\psi$$

- Mean field approximation: interpretation as **single macroscopic wave function** of Bose-Einstein condensate with density $\rho = m|\psi|^2$
- **“FDM equations”** are the **nonlinear Schrödinger–Poisson system of equations**:

$$i\hbar\partial_t\psi_\mathrm{c} = -\frac{\hbar^2}{2ma^2}\nabla_\mathrm{c}^2\psi_\mathrm{c} + \frac{m}{a}\Phi_\mathrm{c}\psi_\mathrm{c}$$
$$\nabla_\mathrm{c}^2\Phi_\mathrm{c} = 4\pi G m\left(|\psi_\mathrm{c}|^2 - \langle|\psi_\mathrm{c}|^2\rangle\right)$$

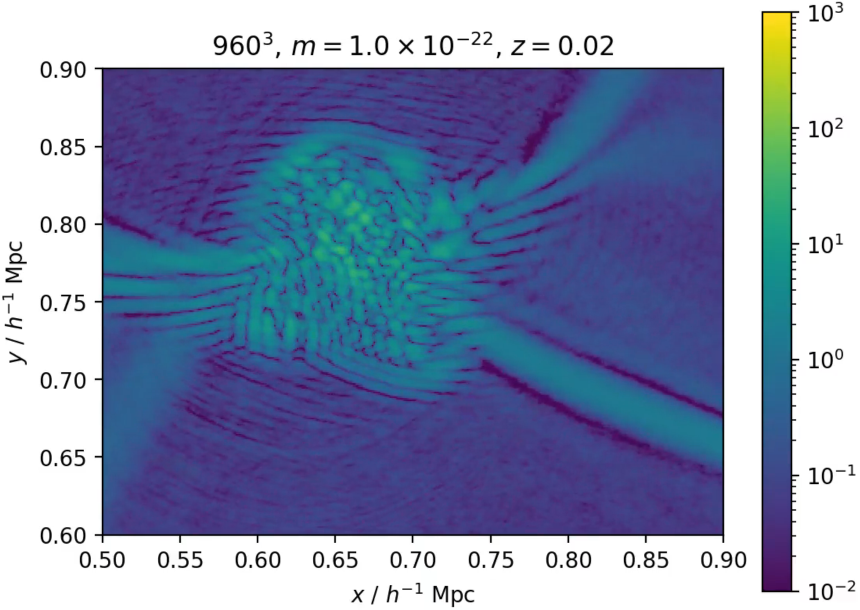
Only a single scale, determined by $\frac{\hbar}{m}$ (→ wavelength)

Impact of fuzzy dark matter – #1: Initial conditions



- Can be computed using `AxionCAMB`, Hlozek et al. (2015)
- Seeds of structure suppressed below quantum wavelength

Impact of fuzzy dark matter – #2: Dynamics/structure



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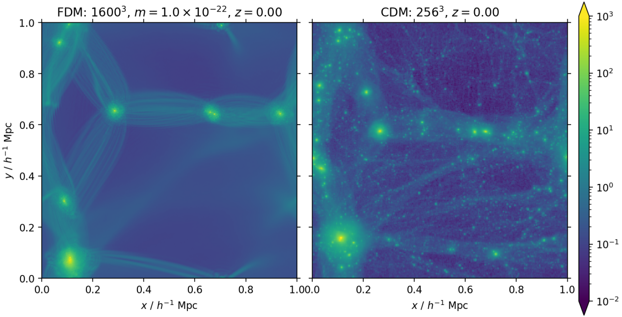
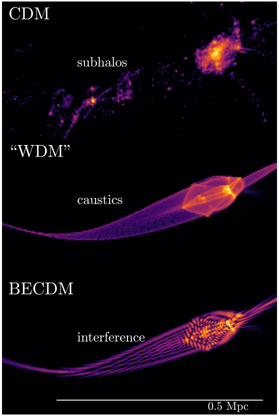
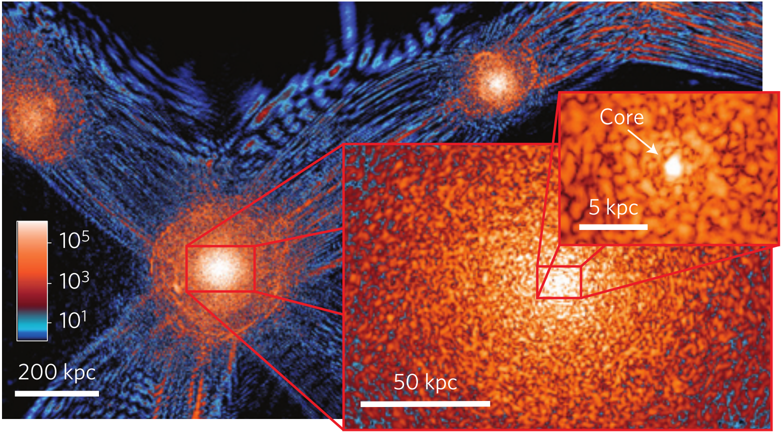




Mocz, Fialkov, et al. (2020)

- Quantum fluctuations, interference patterns
- Small-scale (sub-)structure suppressed

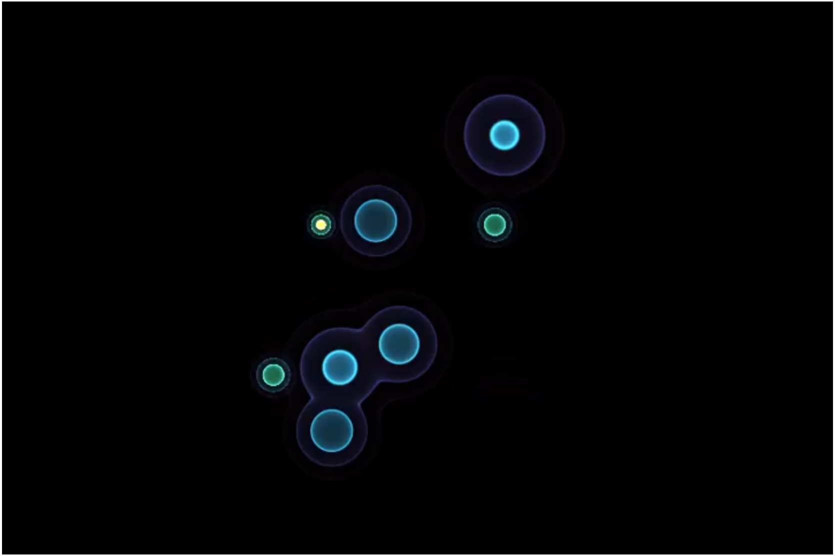
Impact of fuzzy dark matter – #2: Dynamics/objects (solitons)



Schive, Chiueh, and Broadhurst (2014)

- Ground state of FDM equations: **soliton** (spherical "**core**")
- Soliton(-like) cores form at the center of all virialized halos
- Fluctuations around & within soliton cores ⇒ **dynamical heating** (e. g. of stars)
- $M_c \sim \frac{1}{m^2 r_c}$ ⇒ cores are smaller for larger core or particle mass

Impact of fuzzy dark matter – #2: Dynamics/objects (solitons)



Mocz, Vogelsberger, et al. (2017)

Numerical approaches to fuzzy dark matter simulations

I. Schrödinger–Poisson equations

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2ma^2}\nabla^2\psi + \frac{m}{a}\Phi\psi$$
$$\nabla^2\Phi = 4\pi G m(|\psi|^2 - \langle|\psi|^2\rangle)$$

II. Madelung formulation (fluid dynamics representation)

$$\partial_t\rho + \nabla\cdot\rho\vec{v} = 0$$
$$\partial_t\vec{v} + \frac{1}{a^2}(\vec{v}\cdot\nabla)\vec{v} = -\nabla\left(\frac{1}{a}\Phi - \underbrace{\frac{\hbar^2}{2m^2a^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}}_{=Q}\right)$$
$$\nabla^2\Phi = 4\pi G(\rho - \bar{\rho})$$

$$\psi = \sqrt{\frac{\rho}{m}}e^{i\alpha}$$
$$\rho = m|\psi|^2$$
$$\vec{v} = \frac{\hbar}{m}\nabla\alpha$$

- Phase is undefined for $\rho = 0$
  $\Rightarrow$ significant effects on overall evolution

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"quantum potential"
"quantum pressure"

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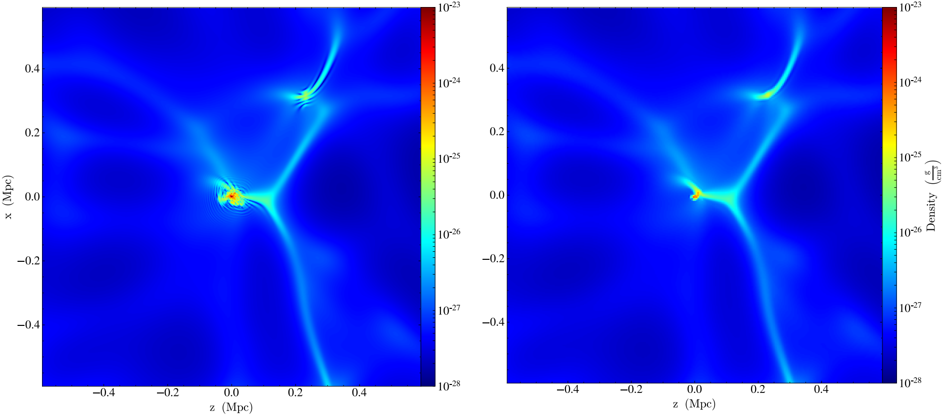
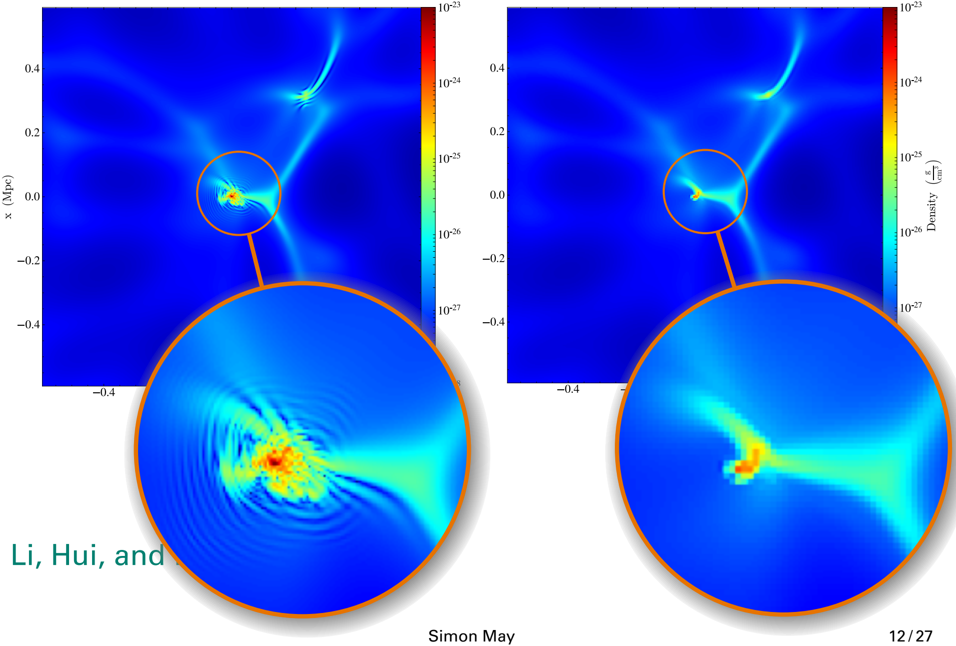
Schrödinger–Poisson vs. Madelung formulation



Li, Hui, and Bryan (2018)

Schrödinger–Poisson vs. Madelung formulation



Li, Hui, and

Using pseudo-spectral methods to simulate fuzzy dark matter

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2ma^2}\nabla^2\psi + \frac{m}{a}\Phi\psi$$
$$\nabla^2\Phi = 4\pi G m(|\psi|^2 - \langle|\psi|^2\rangle)$$

- Symmetrized split-step Fourier method ("kick–drift–kick")
- Algorithm:

$$\psi \leftarrow e^{-i\frac{m}{\hbar}\frac{\Delta t}{2}\Phi}\psi \qquad \text{kick}$$
$$\psi \leftarrow \mathrm{IFFT}\Big(e^{-i\frac{\hbar}{m}\frac{\Delta t}{2}k^2}\,\mathrm{FFT}(\psi)\Big) \qquad \text{drift}$$
$$\Phi \leftarrow \mathrm{IFFT}\Big(-\frac{1}{k^2}\,\mathrm{FFT}\big(4\pi G m(|\psi|^2 - \langle|\psi|^2\rangle)\big)\Big) \qquad \text{update potential}$$
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Choice of time step: $\Delta t < \min\Big(\frac{4}{9\pi}\frac{m}{\hbar}a^2\Delta x^2\ ,\ 2\pi\frac{\hbar}{m}a\frac{1}{|\Phi_{\max}|}\Big)$

- "Exact" solution
- Automatic conservation of mass
- Can adapt existing particle–mesh (PM) code
- Simple implementation

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Correspondence of CDM and FDM initial conditions

- Cold dark matter (CDM): Collisionless fluid, described by a phase space distribution function $f(\vec{x}, \vec{v})$
- Can construct a wave function ψ from a distribution function f:

$$\psi(\vec{x}) \sim \sum_{\vec{v}} \sqrt{f(\vec{x}, \vec{v})} e^{im/\hbar\, \vec{x}\cdot\vec{v} + R_{\vec{v}}}$$

  For "cold"/single-stream distribution function:

$$\psi = \sqrt{\frac{\rho}{m}} e^{i\alpha}$$

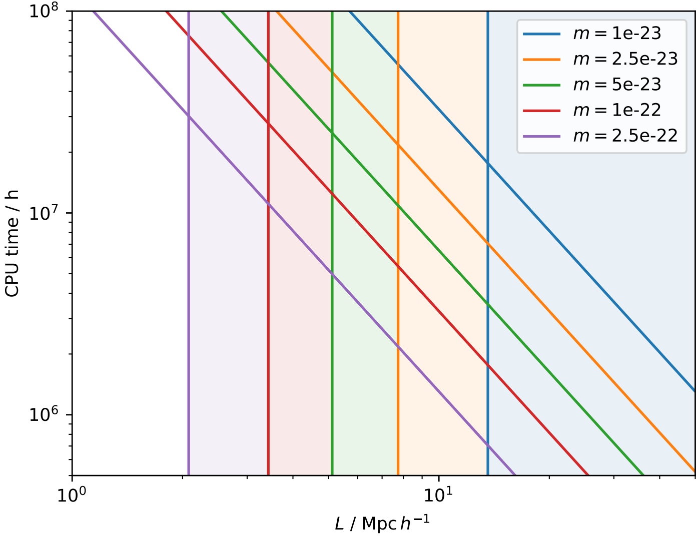
$$\vec{v} = \frac{\hbar}{m} \nabla\alpha$$

- Grid discretization implies a maximum velocity which can be represented
  Mocz, Lancaster, et al. (2018), Mocz, Fialkov, et al. (2020)

$$v < \frac{\hbar}{m} \frac{\pi}{\Delta x}$$

Computational cost of fuzzy dark matter simulations

Computational cost scales as $\sim m^{-1}$ with mass, $\sim L^{-2}$ with box size & $\sim N^5$ with grid size!



8192^3 grid

10^7 CPU h $\triangleq$ two full months of entire MPA cluster (FREYA)!

$$\Delta t < \frac{4}{9\pi} \frac{m}{\hbar} a^2 \Delta x^2$$

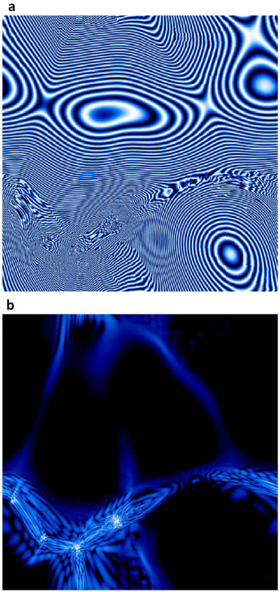
$$v < \frac{\hbar}{m} \frac{\pi}{\Delta x}$$

Why is it hard to simulate fuzzy dark matter?

Computational challenges

- **Both** large scales and small (kpc-scale) de Broglie wavelength must be resolved for correct evolution
- High velocities require high resolution even in low-density regions
  Velocity criterion: $v < \frac{\hbar}{m} \frac{\pi}{\Delta x}$
- **Time step criterion:** $\Delta t \sim \Delta x^2$
  (seems to be approach-independent)

- (Tooling: Hydrodynamics codes are designed for N-body simulations)

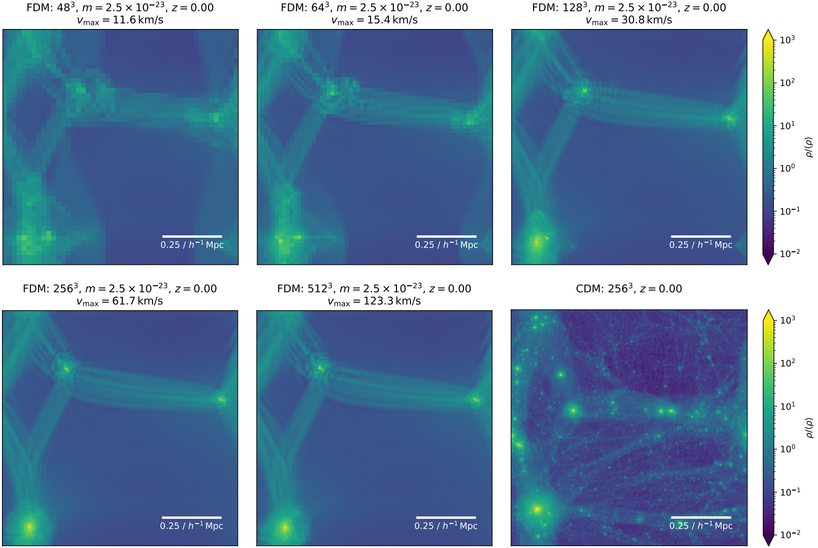


Schive, Chiueh, and Broadhurst (2014)

Resolution effects

Impact of lacking resolution

1 Mpc box projections; 99th-percentile velocity in the CDM simulation is 97.2 km/s

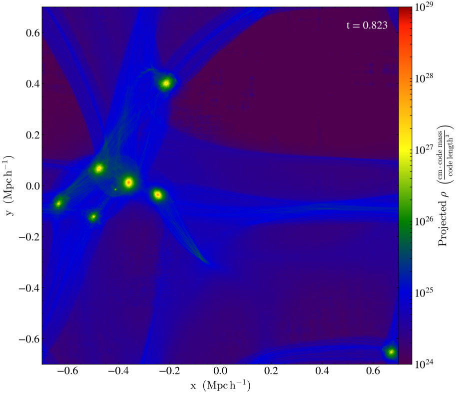


Not immediately obvious that something went wrong!

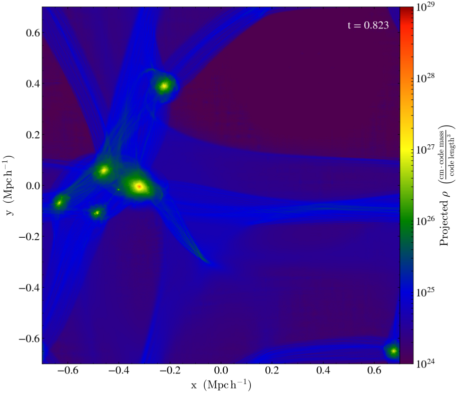
Resolution effects

Halo formation

Lower resolution



Higher resolution

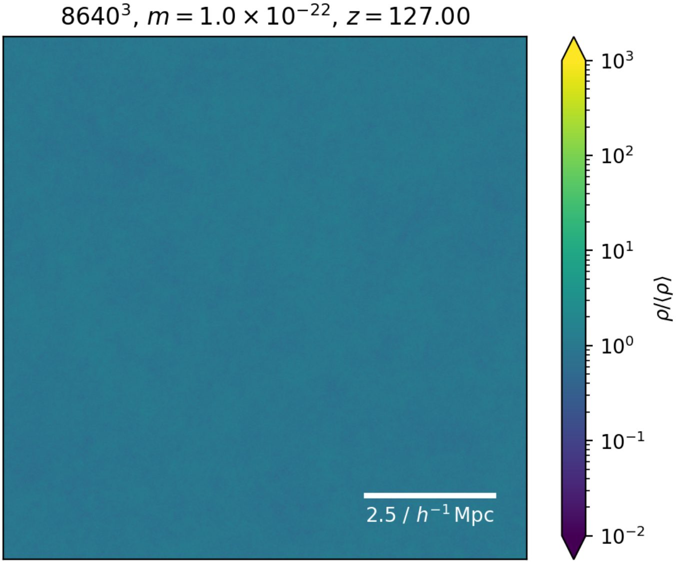


Evolution is delayed/halted for simulations with insufficient resolution

(images from Schive, FDM Workshop Göttingen 2020)

Largest cosmological FDM simulation so far

8640^3 grid, 10 Mpc/h box, $m =$ 10^{-22} eV (slices through simulation volume)



May et al., in prep.

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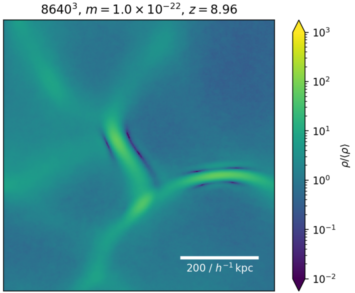
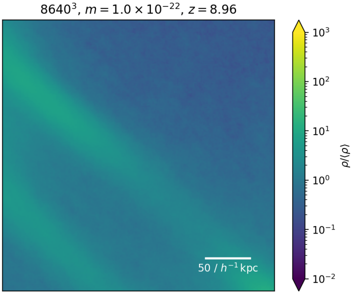
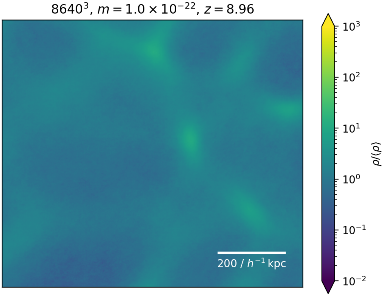
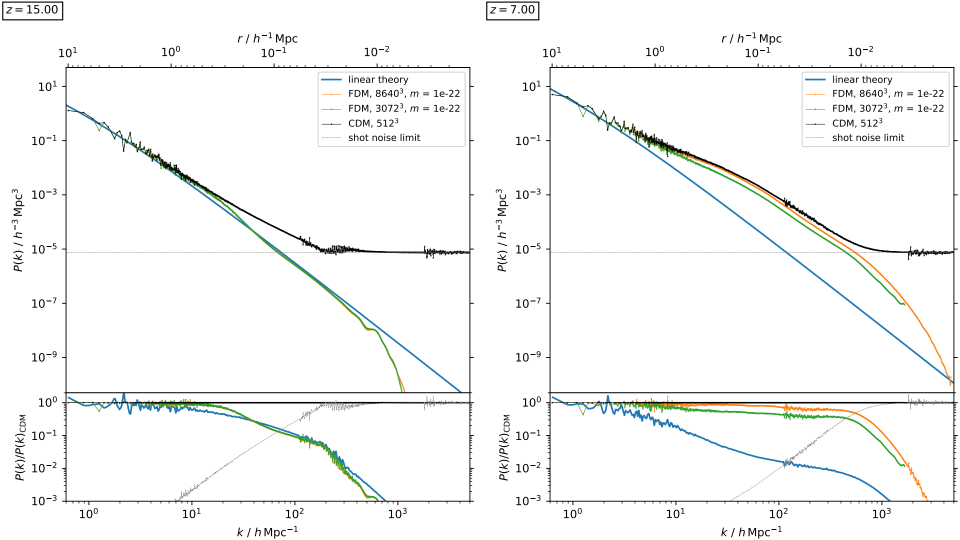






First results: Dark matter power spectrum

10 Mpc/h box simulations



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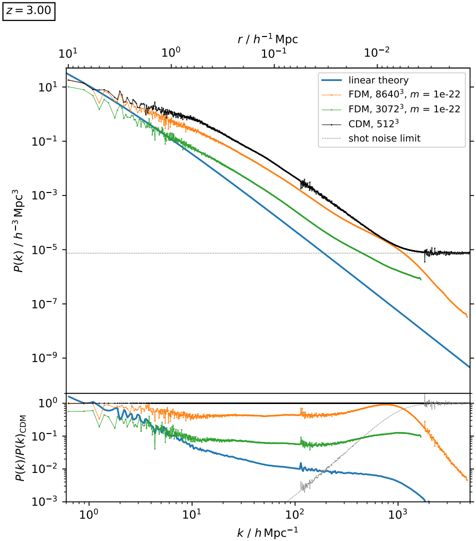
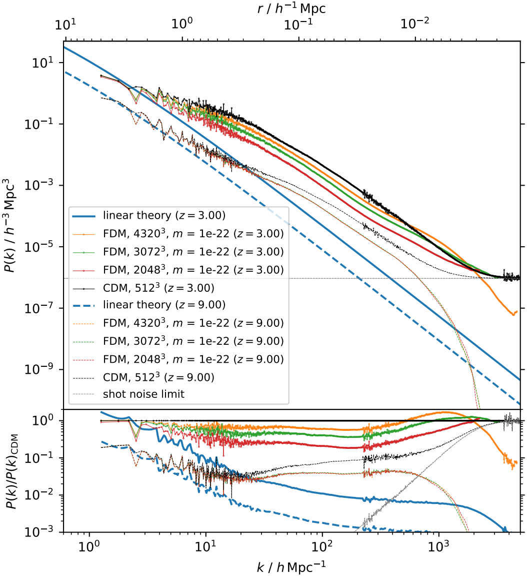


- Agreement on large scales
- Delayed structure formation
- Suppression on small scales
- Insufficient resolution causes structure to “freeze” below a certain redshift

May et al., in prep.

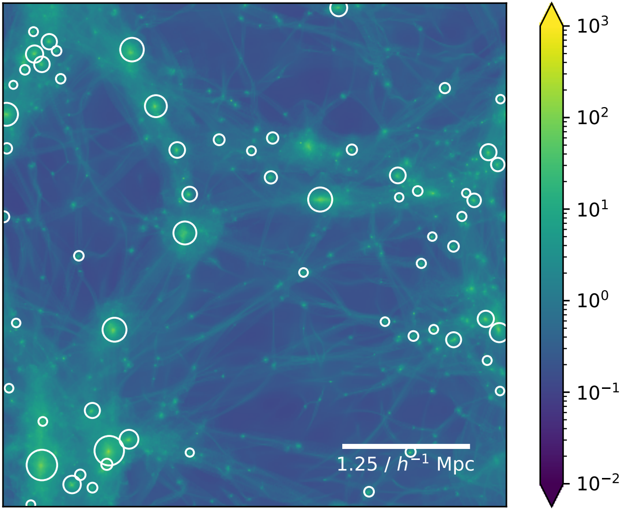
First results: Dark matter power spectrum

5 $^{\mathrm{Mpc}}/_{h}$ box simulations



May et al., in prep.

Finding halos

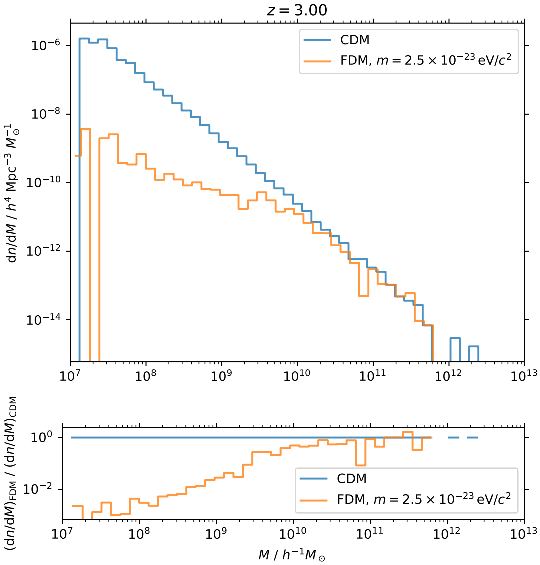


May et al., in prep.

Developed modified friends-of-friends halo finder to operate on a Cartesian grid

First results: Halo mass function

10 Mpc/h box simulations



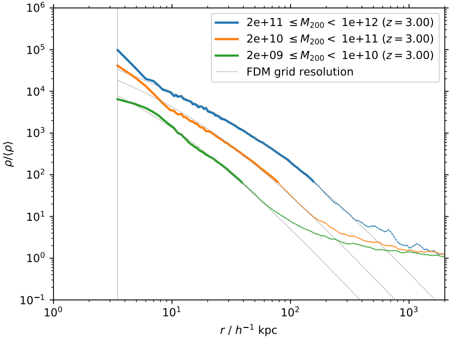
- Fewer low-mass objects
- Similar counts for the largest halos

May et al., in prep.

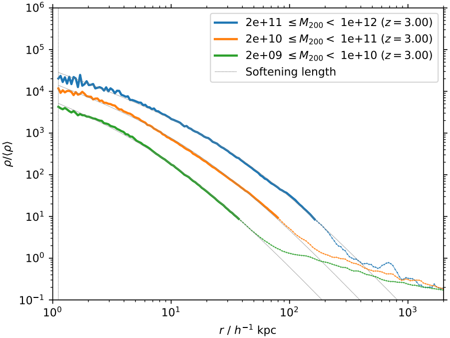
First results: Dark matter halo profiles (stacked)

10 Mpc/h box simulations

FDM ($m = 2.5 \times 10^{-23}$ eV/c²)



CDM



- Fuzzy dark matter halos form central cores with flat density profiles
- Adaptive resolution better suited to show inner halos

May et al., in prep.

Summary

Main problems for fuzzy dark matter simulations:

1. **Time integration** $\Delta t \sim \Delta x^2$
2. **Rapid oscillations** even in low-density regions
3. **Large dynamic range**: "large"-scale structure simulations must still resolve de Broglie wavelength, limited to $\lesssim 10\,\mathrm{Mpc}/h$ box
4. "New" field without decades of experience or refined codes/methods as for CDM

Features of FDM dynamics compared to CDM:

1. Modified **initial power spectrum**, small scales suppressed
2. **Suppression** of structure below de Broglie wavelength ($\approx$ kpc) $\rightarrow$ Heisenberg uncertainty principle
3. Formation of **halo cores**
4. Fluctuating $\approx$ kpc quantum **interference patterns**

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