# CMB lensing by Jaiseung Kim

CMB intensity

$$I(\hat{n},\nu) \approx B(\nu, T_0) + \left.\frac{\partial B(\nu,T)}{\partial T}\right|_{T=T_0} \Delta T(\hat{n}, \nu)$$

$\hat{n}$: sky direction

$\nu$: frequency

$B(\nu, T_0)$ : Plank function

$T_0 = 2.725 \pm 0.002$ K

$$\frac{\Delta T}{T_0} \sim 10^{-5}.$$

CMB anisotropy is expanded in terms of spherical harmonics:

$$T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

On a sphere, spherical harmonics are orthonormal.

$$\int Y_{lm}(\hat{n}) \, Y^{*}_{l'm'}(\hat{n}) \, d\Omega = \delta_{ll'} \delta_{mm'}$$

acoustic oscillations in the primordial photon-baryon fluids with initial seed fluctuation of adiabatic Gaussian, nearly scale invariant spectrum.

② 

statistical isotropy of our early Universe requires :

$$\langle a_{\ell m} \rangle = 0$$

$$\langle a_{\ell m} \, a_{\ell' m'}^{*} \rangle = \delta_{\ell \ell'} \, \delta_{m m'} \, C_\ell .$$

$\langle \cdots \rangle$ : ensemble average

Angular correlation of CMB anisotropy is

$$\langle T(\hat{n}) T(\hat{n}') \rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell (\cos\theta)$$

$P_\ell$ : Legendre polynomials

$\theta = \cos^{-1} (\hat{n} \cdot \hat{n}')$

$$P_\ell (\cos\theta) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}) \, Y_{\ell m}^{*}(\hat{n}')$$

③

# CMB lensing.

r.m.s total deflection ~ 2 arcminutes

comparable to angular scales $\ell \gtrsim 3000$.

small angle approximation.

$\Uparrow$

clusters : ~ arcminute deflection.

galaxies : ~ arcsecond deflection.

very small fraction of lines of sight encounter dense bodies (e.g. black holes)

strong lensing and significant magnifications

: sub-arcminute scales.

Born approximation :

calculating lensing effect along the unperturbed path

# Perturbed Photon Path

$$\frac{d^2\chi}{d\eta^2} + 2\frac{d\Psi}{d\eta} = 0.$$

$$\frac{d^2\theta}{d\eta^2} - \frac{2}{\chi}\frac{d\theta}{d\eta} + \frac{2}{\chi^2}\frac{\partial\Psi}{\partial\theta} = 0.$$

$$\frac{d^2\phi}{d\eta^2} - \frac{2}{\chi}\frac{d\phi}{d\eta} + \frac{2}{\chi^2\sin^2\theta}\frac{\partial\Psi}{\partial\phi} = 0,$$

$\chi$: comoving distance.

$\eta$: Conformal time.

$$\chi = \eta_0 - \eta - 2\int_{\eta_0}^{\eta} \Psi\, d\eta'$$

$\chi \simeq \eta_0 - \eta$

Born approxim

$$\theta = \theta_0 - \int_0^{\chi_*} d\chi\, \frac{\chi_* - \chi}{\chi_*\chi}\, 2\frac{\partial}{\partial\theta}\Psi$$

$$\phi = \phi_0 - \int_0^{\chi_*} d\chi\, \frac{\chi_* - \chi}{\chi_*\chi}\, \frac{2}{\sin^2\theta}\frac{\partial}{\partial\phi}\Psi.$$

④ deflection.

$$\alpha = -2\int_0^{\chi_*} d\chi \quad \frac{\chi_* - \chi}{\chi_* \chi} \quad \nabla_{\hat{n}} \Phi.$$

$$\alpha_\theta = \theta - \theta_0$$

$$\alpha_\phi = \sin^2\theta \quad (\phi - \phi_0)$$

magnification matrix

$$A_{ij} = \delta_{ij} + \partial_i \alpha_j$$

$$= \delta_{ij} - 2\int_0^{\chi_*} d\chi \quad \frac{\chi_* - \chi}{\chi_* \chi} \quad \partial_i \partial_j \Phi.$$

$$= \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 + w \\ -\gamma_2 - w & 1 - \kappa + \gamma_1 \end{pmatrix}.$$

⑤ lensing power spectrum.

projected lensing potential

$$\psi(\hat{n}) = -2 \int_0^{\chi_*} d\chi \frac{\chi_* - \chi}{\chi_* \chi} \Psi.$$

projected lensing potential

Spherical harmonic expansion

$$\psi(\hat{n}) = \sum_{lm} \psi_{lm} Y_{lm}(\hat{n}).$$

$$\langle \psi_{lm} \psi_{lm}^* \rangle = \delta_{ll'} \delta_{mm'} C_l^\psi$$

$$\langle \psi(\hat{n}) \psi(\hat{n}') \rangle = \frac{1}{4\pi} \sum_l (2l+1) C_l^\psi P_l(\hat{n} \cdot \hat{n}').$$

Fourier Expansion in 3D space.

$$\Psi(x;\eta) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \Psi(k;\eta) e^{ik\cdot x}.$$

$$\langle \Psi(k;\eta) \Psi^*(k';\eta') \rangle = \frac{2\pi^2}{k^3} P_\Psi(k) \delta(k-k') \underline{T(k,\eta)}\ \underline{T(k',\eta')}$$

transfer function.

⑥

$\langle \psi(\hat{n}) \psi(\hat{n}') \rangle =$

$$4 \int_0^{\chi_*} d\chi \int_0^{\chi_*} d\chi' \left( \frac{\chi_* - \chi}{\chi_* \chi} \right) \left( \frac{\chi_* - \chi'}{\chi_* \chi'} \right)$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{2\pi^2}{k^3} P_{\mathcal{I}}(k) \, T(k, \eta) \, T(k', \eta') \, e^{i k \cdot x} \, e^{-i k \cdot x'}$$

$$e^{i k x} = 4\pi \sum_{\ell m} i^\ell j_\ell(k\chi) \, Y_{\ell m}^*(\hat{n}) \, Y_{\ell m}(\hat{k})$$

$$C_\ell^\psi = 16\pi \int \frac{dk}{k} P_{\mathcal{I}}(k) \left[ \int_0^{\chi_*} d\chi \, T(k, \eta_0 - \chi) \, j_\ell(k\chi) \left( \frac{\chi_* - \chi}{\chi_* \chi} \right) \right]^2.$$

For small scales, flat sky approximation.

$$\langle T(x) \, T(x') \rangle = \xi(|x - x'|)$$

$$C_\ell^{TT} = 2\pi \int_0^\infty dr \, r \, J_0(r\ell) \, \xi(r)$$

⑦

Flat sky approximation

$\tilde{T}(x) = T(x+\alpha)$ $\qquad\qquad \alpha = \nabla\psi$

the lensed correlation

$$\tilde{\xi}(r) = \langle \tilde{T}(x)\, \tilde{T}(x') \rangle = \langle T(x+\alpha)\, T(x'+\alpha') \rangle$$

CMB and deflection angle are virtually uncorrelated.

$$= \int \frac{d^2 l}{2\pi} \int \frac{d^2 l'}{2\pi} \langle e^{-i l\cdot(x+\alpha)}\, e^{i l'\cdot(x'+\alpha')} \rangle$$

$$\langle T(l)\, T^*(l') \rangle$$

$r = x - x'$

$$= \int \frac{d^2 l}{(2\pi)^2} C_l\, e^{-i l\cdot r} \langle e^{i l\cdot(\alpha'-\alpha)} \rangle$$

$$\langle e^{i l\cdot(\alpha'-\alpha)} \rangle = \exp\left(-\frac{1}{2}\langle [l\cdot(\alpha'-\alpha)]^2 \rangle\right)$$

$$= \exp\left(-\frac{1}{2} l^2 [A_0(0) - A_0(r) + A_2(r)\cos 2\phi]\right).$$

$\phi = \cos^{-1}(l\cdot r).$

⑧

$$\tilde{C}_\ell = \frac{1}{4\pi}\int_0^\infty r\,dr \quad J_0(r\ell)\,\tilde{\xi}(r)$$

Powerspectrum of lensed CMB

angular correlation of lensed CMB

$$\tilde{\xi}(r) = \int \frac{d^2\ell}{(2\pi)^2}\, C_\ell \exp[-i\ell r\cos\phi]\exp\left[-\frac{\ell^2}{2}\left[A_0(0) - A_0(r) + A_2(r)\cos 2\phi\right]\right]$$

$$A_0(r) = \int_0^\infty \frac{d\ell\,\ell^3}{2\pi}\, C_\ell^\psi\, J_0(r\ell)$$

$$A_2(r) = \int_0^\infty \frac{d\ell\,\ell^3}{2\pi}\, C_\ell^\psi\, J_2(r\ell).$$

(9) small angle approximation

lensed CMB Temperature

$$\tilde{T}(\hat{n}) = T(\hat{n}+\alpha) = T(\hat{n}+\nabla\psi)$$

$$\simeq T(\hat{n}) + \nabla\psi(\hat{n})\cdot\nabla T(\hat{n}) + O(\psi^2)$$

$$\tilde{a}_{\ell m} \simeq a_{\ell m} + \sum_{LM}\sum_{\ell' m'} \psi_{LM}\, a_{\ell' m'}\, I_{\ell}^{m}{}_{L}^{M}{}_{\ell'}^{m'}$$

$$(-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \begin{pmatrix} \ell & L & \ell' \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times \sqrt{\frac{(2L+1)(2\ell+1)(2\ell'+1)}{16\pi}} \left[ L(L+1) + \ell'(\ell'+1) - \ell(\ell+1) \right]$$

$$\langle \tilde{a}_{\ell m}\, \tilde{a}^{*}_{\ell' m'} \rangle \propto \delta_{\ell\ell'}\,\delta_{mm'}$$

$\times$

Violation of statistical isotropy

$\frac{\ell(\ell+1)}{2\pi} C_\ell^{TT}$

# lensed CMB Polarization.

$$\tilde{Q}(\hat{n}) + i\tilde{U}(\hat{n}) = Q(\hat{n}+\alpha) + iU(\hat{n}+\alpha)$$

$$= Q(\hat{n}+\nabla\psi) + iU(\hat{n}+\nabla\psi)$$

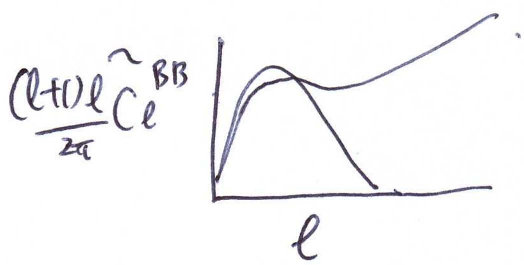
$$\simeq Q(\hat{n}) + iU(\hat{n}) + \nabla\psi(\hat{n})\cdot\nabla\left(Q(\hat{n}) + iU(\hat{n})\right)$$

$$\tilde{a}^{E(B)}_{\ell m} \simeq a^{E(B)}_{\ell m} + \sum_{LM}\sum_{\ell' m'} \psi_{LM} (-1)^m \begin{pmatrix} \ell & \ell' & L \\ m & -m' & -M \end{pmatrix}$$

$$\times \left( \frac{1+(-1)^{L+\ell+\ell'}}{2} a^{E(B)}_{\ell m} + \frac{1-(-1)^{L+\ell+\ell'}}{2i} a^{B(E)}_{\ell m} \right)$$

$$\times \left[ L(L+1) + \ell'(\ell+1) - \ell(\ell+1) \right] \sqrt{\frac{(2L+1)(2\ell+1)(2\ell'+1)}{16\pi}} \begin{pmatrix} \ell & L & \ell' \\ 2 & 0 & -2 \end{pmatrix}$$

$$\langle \tilde{a}^{E}_{\ell m} \tilde{a}^{B*}_{\ell m} \rangle \neq 0 .$$




Lensing produces artificial EB correlation.

(11)

# Lensing Reconstruction

i) $$\langle a^T_{\ell m} \, a^{T*}_{\ell' m'} \rangle \simeq \sum_{LM} (-1)^{M+m'} \begin{pmatrix} \ell & \ell' & L \\ m & -m' & -M \end{pmatrix} \psi_{LM}$$

$$\times \left( C^{TT}_{\ell_1} \, {}_0F_{\ell_2 L \ell_1} + C^{TT}_{\ell_2} \, {}_0F_{\ell_1 L \ell_2} \right).$$

ii) $$\langle a^E_{\ell m} \, a^{B*}_{\ell' m'} \rangle \simeq \sum_{LM} (-1)^{M+m'} \begin{pmatrix} \ell & \ell' & L \\ m & -m' & -M \end{pmatrix} \psi_{LM}$$

$$\times i \left[ C^{EE}_{\ell_1} \, {}_2F_{\ell_2 L \ell_1} - C^{BB}_{\ell_2} \, {}_2F_{\ell_1 L \ell_2} \right]$$

$${}_sF_{\ell L \ell'} = [L(L+1) + \ell'(\ell'+1) - \ell(\ell+1)] \sqrt{\frac{(2L+1)(2\ell+1)(2\ell'+1)}{16\pi}}$$

$$\times \begin{pmatrix} \ell & L & \ell' \\ 2 & 0 & -2 \end{pmatrix}.$$

→ normalization.

linear weight

$$\hat{\psi}_{LM} = \frac{A^\alpha_L}{\sqrt{L(L+1)}} \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} (-1)^M \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix} g^\alpha_{\ell_1 \ell_2}(L) \, a_{\ell_1 m_1} a_{\ell_2 m_2}$$

## ⑫ Integrated Sachs–Wolf effect and Lensing

ISW

$$\Delta T_{ISW}(\hat{n}) = 2\int_0^{\chi_*} d\chi \, \frac{\partial \Phi}{\partial \eta}(\chi\hat{n};\eta)$$

Lensing

$$\psi(\hat{n}) = -2\int_0^{\chi_*} d\chi \, \frac{\chi_* - \chi}{\chi_* \chi} \, \Phi(\chi\hat{n};\eta)$$

$\chi = \eta_0 - \eta$ : comoving distance between us and a ray

$\eta$ : conformal time.

ISW and lensing is highly correlated ($>0.9$)

$$\langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3} \rangle$$

$$\propto \; - C_{l}^{T\psi} C_{l_2}^{T\nabla T} + \cdots$$

produces non-zero 3 point correlation.