13.1.1 Thermodynamic Inference: Lognormal Poisson Model

- $\mathcal{P}(s) = \mathcal{G}(s, S)$
- $\lambda(s) = \kappa\, e^{s}$
- $\mathcal{P}(d^x|\lambda^x) = \frac{(\lambda^x)^{d^x} e^{-\lambda^x}}{d^x!}$

$$\Rightarrow \mathcal{H}(d, s) \quad \widehat{=} \quad \frac{1}{2} s^{\dagger} S^{-1} s - d^{\dagger} s + \kappa^{\dagger} e^{s}$$

$$\widetilde{U}(m, D) \quad = \quad \langle \mathcal{H}(d, s) \rangle_{\mathcal{G}(s-m, D)}$$

13.1.1 Thermodynamic Inference: Lognormal Poisson Model

$$\mathcal{H}(d,\, s) \;\widehat{=}\; \frac{1}{2} s^\dagger S^{-1} s - d^\dagger s + \kappa^\dagger e^s$$

$$\widetilde{U}\,(m,\, D) = \langle \mathcal{H}(d,\, s) \rangle_{\mathcal{G}(s-m,\, D)}$$

$$= \frac{1}{2} \langle s^\dagger S^{-1} s \rangle_{\mathcal{G}(s-m,\, D)} - d^\dagger \langle s \rangle_{\mathcal{G}(s-m,\, D)} + \kappa^\dagger \langle e^s \rangle_{\mathcal{G}(s-m,\, D)}$$

$$\begin{aligned}
\langle s^\dagger S^{-1} s \rangle_{\mathcal{G}(s-m,\, D)} &= \mathrm{Tr}(S^{-1} \langle s s^\dagger \rangle_{\mathcal{G}(s-m,\, D)}) \\
&= \mathrm{Tr}(S^{-1} \langle (m+\varphi)(m+\varphi)^\dagger \rangle_{\mathcal{G}(\varphi,\, D)}) \\
&= \mathrm{Tr}(S^{-1} (m m^\dagger + D)) \\
&= m^\dagger S^{-1} m + \mathrm{Tr}(S^{-1} D)
\end{aligned}$$

13.1.1 Thermodynamical Inference: Lognormal Poisson Model

$$\widetilde{U}(m, D) = \frac{1}{2}\langle s^\dagger S^{-1} s\rangle_{\mathcal{G}(s-m,D)} - d^\dagger \langle s\rangle_{\mathcal{G}(s-m,D)} + \kappa^\dagger \langle e^s\rangle_{\mathcal{G}(s-m,D)}$$

$$\begin{aligned}
\langle s\rangle_{\mathcal{G}(s-m,D)} &= m \\
\langle e^{s^x}\rangle_{\mathcal{G}(s-m,D)} &= \int \mathcal{D}\varphi\, \mathcal{G}(\varphi, D) e^{m^x+\varphi^x} \\
&= e^{m^x} \int \mathcal{D}\varphi\, \frac{\exp\left(-\frac{1}{2}\varphi^\dagger D^{-1}\varphi + j^\dagger\varphi\right)}{|2\pi D|^{1/2}} \quad \text{with } j_y = \delta(y-x) \\
&= e^{m^x} e^{\frac{1}{2} j^\dagger D j} \\
&= e^{m^x + \frac{1}{2} D^{xx}}
\end{aligned}$$

13.1.1 Thermodynamic Inference: Lognormal Poisson Model

$$
\begin{aligned}
\Rightarrow \widetilde{U}(m, D) &= \frac{1}{2} m^\dagger S^{-1} m + \frac{1}{2} \mathrm{Tr}(D S^{-1}) - d^\dagger m + \kappa^\dagger e^{m + \frac{1}{2}\hat{D}} \\
\widetilde{S_B}(D) &= \frac{1}{2} \mathrm{Tr}(1 + \ln(2\pi D)) \\
\widetilde{G}(m, D) &= \widetilde{U}(m, D) - T\,\widetilde{S_B}(D) \\
&= \frac{1}{2} m^\dagger S^{-1} m + \frac{1}{2} \mathrm{Tr}(D S^{-1}) - d^\dagger m + \kappa^\dagger e^{m + \frac{1}{2}\hat{D}} - \frac{T}{2} \mathrm{Tr}(1 + \ln(2\pi D))
\end{aligned}
$$

13.1.1 Thermodynamical Inference: Lognormal Poisson Model

$$\widetilde{G}(m, D) = \frac{1}{2} m^\dagger S^{-1} m + \frac{1}{2} \mathrm{Tr}(D S^{-1}) - d^\dagger m + \kappa^\dagger e^{m + \frac{1}{2}\hat{D}} - \frac{T}{2} \mathrm{Tr}(1 + \ln(2\pi D))$$

Mean map:

$$\frac{\delta \widetilde{G}(m, D)}{\delta m} = S^{-1} m - d + \kappa\, e^{m + \frac{1}{2}\hat{D}} \stackrel{!}{=} 0$$

$$\Rightarrow m = S(d - \kappa\, e^{m + \frac{1}{2}\hat{D}})$$

Uncertainty dispersion:

$$D = T \left(\frac{\delta^2 G}{\delta m \delta m^\dagger} \right)^{-1} = T \left(\frac{\delta}{\delta m} (S^{-1} m - d + \kappa\, e^{m + \frac{1}{2}\hat{D}}) \right)^{-1} = T \left(S^{-1} + \widehat{\kappa e^{m + \frac{1}{2}\hat{D}}} \right)^{-1}$$

13.1.2 Gibbs Free Energy & Variational Inference

$$
\begin{aligned}
\widetilde{G}(m, D) &= \langle \underbrace{\mathcal{H}(d, s)}_{=-\ln \mathcal{P}(d,s)} + \underbrace{\ln \mathcal{G}(s-m, D)}_{=-S_B} \rangle_{\mathcal{G}(s-m, D)} \\
&= \int \mathcal{D}s\, \mathcal{G}(s-m, D) \ln \frac{\mathcal{G}(s-m, D)}{\mathcal{P}(d, s)} \\
&= \int \mathcal{D}s\, \mathcal{G}(s-m, D) \ln \frac{\mathcal{G}(s-m, D)}{\mathcal{P}(s|d)} - \ln \mathcal{P}(d) \\
&\widehat{=} \int \mathcal{D}s\, \mathcal{G}(s-m, D) \ln \frac{\mathcal{G}(s-m, D)}{\mathcal{P}(s|d)} \\
&= \int \mathcal{D}s\, \widetilde{\mathcal{P}}(s|\widetilde{d}) \ln \frac{\widetilde{\mathcal{P}}(s|\widetilde{d})}{\mathcal{P}(s|d)}, \text{ with } \widetilde{d} = (m, D) \\
&= D_{\mathrm{KL}}(\widetilde{\mathcal{P}}(s|\widetilde{d}) || \mathcal{P}(s|d))
\end{aligned}
$$

13.1.2 Gibbs Free Energy & Variational Inference

$$\widetilde{G}\,(m, D) \quad \widehat{=} \quad D_{\mathrm{KL}}(\widetilde{\mathcal{P}}(s|\,\widetilde{d})||\mathcal{P}(s|d))$$

minimize $D_{\mathrm{KL}}(\widetilde{\mathcal{P}}(s|\,\widetilde{d})||\mathcal{P}(s|d))$ w.r.t. $\widetilde{d}$
Gibbs free energy minimization = variational inference (VI)

minimize $D_{\mathrm{KL}}(\mathcal{P}(s|d)||\widetilde{\mathcal{P}}(s|\,\widetilde{d}))$ w.r.t. $\widetilde{d}$
Optimal coding = expectation propagation (EP)

13.2 Operator Calculus for Information Field Theory

$$\langle f(s)\rangle_{\mathscr{G}(s-m,D)} \quad = \quad ?$$

$$\langle s\rangle_{\mathscr{G}(s-m,D)} = \int \mathcal{D}s\, s\, \mathscr{G}(s-m,D) = \int \mathcal{D}s\, \frac{s\, e^{(s-m)^\dagger D^{-1}(s-m)}}{|2\pi D|^{\frac{1}{2}}}$$

Observation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}m}\mathscr{G}(s-m,D) \quad &= \quad D^{-1}(s-m)\,\mathscr{G}(s-m,D) \\ \Rightarrow \left(D\frac{\mathrm{d}}{\mathrm{d}m}+m\right)\mathscr{G}(s-m,D) \quad &= \quad s\,\mathscr{G}(s-m,D) \end{aligned}$$

13.2 Operator Calculus for Information Field Theory

$$\left(D\frac{\mathrm{d}}{\mathrm{d}m} + m\right)\mathcal{G}(s-m,D) \;=\; s\,\mathcal{G}(s-m,D)$$

$$\begin{aligned}
\Rightarrow \langle s\rangle_{\mathcal{G}(s-m,D)} &= \int \mathcal{D}s\,\left(D\frac{\mathrm{d}}{\mathrm{d}m} + m\right)\mathcal{G}(s-m,D)\\
&= \left(D\frac{\mathrm{d}}{\mathrm{d}m} + m\right)\int \mathcal{D}s\,\mathcal{G}(s-m,D)\\
&= \left(D\frac{\mathrm{d}}{\mathrm{d}m} + m\right)\underbrace{\langle 1\rangle_{\mathcal{G}(s-m,D)}}_{=1} = 0 + m\\
\Rightarrow \langle s^n\rangle_{\mathcal{G}(s-m,D)} &= \left(D\frac{\mathrm{d}}{\mathrm{d}m} + m\right)^n 1
\end{aligned}$$

13.2 Operator Calculus for Information Field Theory

- field operator: $\Phi := D\frac{\mathrm{d}}{\mathrm{d}m} + m$
- vacuum vector: $1 : \; m \mapsto 1$
- $f(s) = \sum_{i=0}^{\infty} \lambda_i s^i$

$$
\begin{aligned}
\langle f(s)\rangle_{\mathscr{G}(s-m,D)} &= \sum_{i=0}^{\infty} \lambda_i \left\langle s^i \right\rangle_{\mathscr{G}(s-m,D)} \\
&= \sum_{i=0}^{\infty} \lambda_i \left\langle \Phi^i \right\rangle_{\mathscr{G}(s-m,D)} \\
&= \sum_{i=0}^{\infty} \lambda_i \Phi^i 1 = f(\Phi)\, 1
\end{aligned}
$$

Annihilation and Creation Operator

- field operator: $\Phi := D\frac{\mathrm{d}}{\mathrm{d}m} + m = a + a^{+}$
- annihilation operator: $a := D\frac{\mathrm{d}}{\mathrm{d}m}$, $a^{x} = D^{xy}\ \frac{\mathrm{d}}{\mathrm{d}m^{y}}$
- creation operator: $a^{+} := m$, $a^{+x} = m^{x}$

Canonical commutation relations:

$$
\begin{aligned}
[a^{+x}, a^{+y}]\, X(m) &= (m^{x}m^{y} - m^{y}m^{x})\ X(m) = 0 \\
[a^{x}, a^{y}]\, X(m) &= \left(D^{xz}\ \frac{\mathrm{d}}{\mathrm{d}m^{z}}\, D^{yz'}\ \frac{\mathrm{d}}{\mathrm{d}m^{z'}} - D^{yz'}\ \frac{\mathrm{d}}{\mathrm{d}m^{z'}}\, D^{xz}\ \frac{\mathrm{d}}{\mathrm{d}m^{z}} \right) X(m) \\
&= D^{xz}D^{yz'}\ \underbrace{\left(\frac{\mathrm{d}^{2}}{\mathrm{d}m^{z}\mathrm{d}m^{z'}} - \frac{\mathrm{d}^{2}}{\mathrm{d}m^{z'}\mathrm{d}m^{z}} \right) X(m)}_{=0} = 0
\end{aligned}
$$

Annihilation and Creation Operator

- field operator: $\Phi := D\frac{\mathrm{d}}{\mathrm{d}m} + m = a + a^{+}$
- annihilation operator: $a := D\frac{\mathrm{d}}{\mathrm{d}m}$, $a^x = D^{xy}\,\frac{\mathrm{d}}{\mathrm{d}m^y}$
- creation operator: $a^{+} := m$, $a^{+x} = m^x$

Canonical commutation relations:

$$
\begin{aligned}
[a^x, a^{+y}]\, X(m) &= D^{xz}\,\frac{\mathrm{d}}{\mathrm{d}m^z}\,[m^y\, X(m)] - m^y D^{xz}\,\frac{\mathrm{d}}{\mathrm{d}m^z}\, X(m) \\
&= \left(D^{xz}\,\underbrace{\left[\frac{\mathrm{d}m^y}{\mathrm{d}m^z}\right]}_{\mathbb{1}^y_z} + \underbrace{D^{xz} m^y\,\frac{\mathrm{d}}{\mathrm{d}m^z} - m^y D^{xz}\,\frac{\mathrm{d}}{\mathrm{d}m^z}}_{=0} \right) X(m) = D^{xy}\, X \\
[a^{+x}, a^{+y}] &= [a^x, a^y] = 0,\ [a^x, a^{+y}] = D^{xy}
\end{aligned}
$$

Illustration 1

- $\Phi := D\frac{\mathrm{d}}{\mathrm{d}m} + m = a + a^{+}$
- $a := D\frac{\mathrm{d}}{\mathrm{d}m}$, $a^{x} = D^{xy}\ \frac{\mathrm{d}}{\mathrm{d}m^{y}}$
- $a^{+} := m$, $a^{+x} = m^{x}$
- $1 := 1(m)$

Illustration 1:

$$
\begin{aligned}
\langle s^x s^y \rangle_{\mathscr{G}(s-m,D)} &= \Phi^x \Phi^y 1 \\
&= \left(a^x + a^{+x}\right)\left(a^y + a^{+y}\right) 1 \\
&= \left(a^x a^y + a^{+x} a^y + a^x a^{+y} + a^{+x} a^{+y}\right) 1 \\
&= \left(0 + 0 + a^{+y} a^x + [a^x, a^{+y}] + m^x m^y\right)\ 1 \\
&= D^{xy} + m^x m^y
\end{aligned}
$$

Illustration 2

Illustration 2:

$$\left\langle e^{s^x} \right\rangle_{\mathscr{G}(s-m,D)} = e^{\Phi^x} 1 = e^{a^x + a^{+x}} 1$$

Baker-Campbell-Hausdorff (BCH) formula:

$$e^X Y = \sum_{n=0}^{\infty} [X, Y]_n \, e^X$$

with $[X, Y]_n = [X, [X, Y]_{n-1}]$ and $[X, Y]_0 = Y$

In case $[X, [X, Y]] = 0$:

$$e^X Y = Y \, e^X + [X, Y] \, e^X$$
$$e^{X+Y} = e^X \, e^Y e^{\frac{1}{2}[X,Y]}$$

Illustration 2

- $X = a^x$
- $Y = a^{+y}$

$$
\begin{aligned}
e^{a^x} a^{+y} &= a^{+y} e^{a^x} + [a^x, a^{+y}]\, e^{a^x} = a^{+y} e^{a^x} + D^{xy} e^{a^x} \\
e^{a^x + a^{+y}} &= e^{a^{+y}} e^{a^x} e^{\frac{1}{2}[a^x, a^{+y}]} = e^{a^{+y}} e^{a^x} e^{\frac{1}{2} D^{xy}}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \left\langle e^{s^x} \right\rangle_{\mathscr{G}(s-m,D)} &= e^{a^x + a^{+x}} \\
&= e^{a^{+x}} e^{a^x} e^{\frac{1}{2} D^{xx}}\, 1 \\
&= e^{m^x + \frac{1}{2} D^{xx}} \left(1 + a^x + \frac{1}{2} (a^2)^x + \ldots\right) 1 \\
&= e^{m_x + \frac{1}{2} D_{xx}}
\end{aligned}
$$

Illustration 3

Illustrations 3:

$$
\begin{aligned}
\left\langle e^{s^x} e^{s^y} \right\rangle_{\mathscr{G}(s-m,D)} &= e^{\Phi^x} e^{\Phi^y} 1 \\
&= e^{a^x + a^{+x}} e^{a^y + a^{+y}} 1 \\
&= e^{a^{+x} + \frac{1}{2} D^{xx}} e^{a^x} e^{a^{+y} + \frac{1}{2} D^{yy}} e^{a^y} 1 \\
&= e^{a^{+x} + \frac{1}{2} D^{xx} + \frac{1}{2} D^{yy}} e^{a^x} e^{a^{+y}} 1
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \left[e^{a^x}, e^{a^{+y}} \right] &= e^{a^x} e^{a^{+y}} - e^{a^{+y}} e^{a^x} \\
&= e^{a^x + a^{+y} + \frac{1}{2} D^{xy}} - e^{a^x + a^{+y} - \frac{1}{2} D^{xy}} \\
&= e^{a^x + a^{+y}} \left(e^{\frac{1}{2} D^{xy}} - e^{-\frac{1}{2} D^{xy}} \right) \\
&= e^{a^{+y}} e^{a^x} e^{\frac{1}{2} D^{xy}} \left(e^{\frac{1}{2} D^{xy}} - e^{-\frac{1}{2} D^{xy}} \right) \\
&= e^{a^{+y}} e^{a^x} \left(e^{D^{xy}} - 1 \right)
\end{aligned}
$$

Illustration 3

$$
\begin{aligned}
\Rightarrow [e^{a^x}, e^{a^{+y}}] &= e^{a^{+y}} e^{a^x} \left(e^{D^{xy}} - 1\right) \\
e^{a^x} e^{a^{+y}} &= e^{a^{+y}} e^{a^x} e^{D^{xy}} \\
\left\langle e^{s^x} e^{s^y} \right\rangle_{\mathscr{G}(s-m,D)} &= e^{a^{+x} + \frac{1}{2} D^{xx} + \frac{1}{2} D^{yy}} e^{a^x} e^{a^{+y}} 1 \\
&= e^{a^{+x} + \frac{1}{2} D^{xx}} e^{D^{xy}} e^{a^{+y} + \frac{1}{2} D^{yy}} e^{a^x} 1 \\
&= e^{m^x + \frac{1}{2} D^{xx}} e^{D^{xy}} e^{m^y + \frac{1}{2} D^{yy}}
\end{aligned}
$$

Illustration 4

$$
\begin{aligned}
\left\langle e^{s^x} s^y \right\rangle_{\mathscr{G}(s-m,D)} &= e^{\Phi^x} \Phi^y 1 \\
&= e^{a^x + a^{+x}} \left(a^y + a^{+y}\right) 1 \\
&= e^{a^{+x} + \frac{1}{2} D^{xx}} e^{a^x} a^{+y} 1
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow [a^{+y}, e^{a^x}] &= [a^{+y}, \sum_{n=0}^{\infty} \frac{(a^n)^x}{n!}] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} [a^{+y}, (a^n)^x]
\end{aligned}
$$

Illustration 4

$$\begin{aligned}
\Rightarrow [a^{+y}, e^{a^x}] &= \sum_{n=0}^{\infty} \frac{1}{n!} [a^{+y}, (a^n)^x] \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} n [a^{+y}, a^x] (a^{n-1})^x \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} n (a^{n-1})^x [a^{+y}, a^x] \\
&= -e^{a^x} D^{xy}
\end{aligned}$$

$$\begin{aligned}
e^{a^x} a^{+y} &= \left(a^{+y} + D^{xy}\right) e^{a^x} \\
\left\langle e^{s^x} s^y \right\rangle_{\mathscr{G}(s-m,D)} &= e^{a^{+x} + \frac{1}{2} D^{xx}} \left(a^{+y} + D^{xy}\right) e^{a^x} 1 \\
&= e^{m^x + \frac{1}{2} D^{xx}} \left(m^y + D^{xy}\right)
\end{aligned}$$

Illustration 5

Illustration 5:

$$
\begin{aligned}
\left\langle e^{s^x} e^{s^y} s^z \right\rangle_{\mathscr{G}(s-m,D)} &= e^{\Phi^x} e^{\Phi^y} \Phi^z 1 \\
&= e^{a^x + a^{+x}} e^{a^y + a^{+y}} \left(a^z + a^{+z} \right) 1 \\
&= e^{a^{+x} + \frac{1}{2} D^{xx}} e^{a^x} e^{a^{+y} + \frac{1}{2} D^{yy}} e^{a^y} a^{+z} 1 \\
&= e^{a^{+x} + \frac{1}{2} D^{xx}} e^{D^{xy}} e^{a^{+y} + \frac{1}{2} D^{yy}} (a^{+z} + D^{xz} + D^{yz}) e^{a^x} e^{a^y} 1 \\
&= e^{m^x + \frac{1}{2} D^{xx}} e^{D^{xy}} e^{m^y + \frac{1}{2} D^{yy}} (m^z + D^{xz} + D^{yz})
\end{aligned}
$$