Exam on Information Theory

29.05.2021

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- The exam consists of **five exercises**. Please do check if you received all of them.
- There are more questions than you will be able to solve within the given time.
- Therefore jump to the next questions if you cannot solve one.
- The working time is **120 minutes**.
- Please provide mathematically conclusive derivations.
- Fill out and hand in the next page together with your solutions
- All solutions need to be handwritten.

Question	**Points**
1	____/10
2	____/5
3	____/18
4	____/13
5	____/20
Bonus	
Total	____/66

Grade	

GOOD LUCK!

Question 1 ___/10

The function IfThenElse $f(A,B,C)$ takes three logical statements as its arguments. It is defined as if A is *true*, return B, else C:

$$f(A,B,C) = \begin{cases} B & \text{if } A \text{ is } \textit{true} \\ C & \text{else} \end{cases} . \tag{1}$$

Its truth table is given by

A	B	C	f(A,B,C)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Here 0 stands for *false* and 1 stands for *true*. For example the first line states that if A, B and C are *false*, $f(A,B,C)$ will return *false*. The second line states that if A and B are *false* and C is *true*, $f(A,B,C)$ is true, i. e. $f(A,B,C)$ is *true*, if $\overline{A}\,\overline{B}C$ is *true*.

a) Using Boolean Algebra, show that the truth table coincides with the definition of f in Equation (1). Therefore find all the cases where f is *true* and formulate f within the formalism of Boolean Algebra by logically adding all cases where f is *true*. Bring this expression into the form ___/3

$$f(A,B,C) = AB + \overline{A}C. \tag{2}$$

b) Show that ___/3

$$\overline{f}(A,B,C) := \overline{f(A,B,C)} = \overline{A}\,\overline{C} + A\overline{B} + \overline{B}\,\overline{C} \tag{3}$$

c) Calculate $\overline{f}(f(A,B,C),B,C)$. ___/4

Question 2 ___/5

a) Show that the expected information gain on some signal from some to-be-taken data is the mutual information of data and signal. ___/5

Question 3 ___/18

You take part in a gameshow. At one point in the show, the host presents you **four** doors, each hiding one prize. You get to choose one of the doors and get to keep whatever is behind it. **Three** of the doors hide a goat, and one is hiding a sportscar. Initially, you choose a door. Then the game has two rounds:

1. The host, who knows which door is hiding the car, opens one of the doors you have not chosen, making sure he reveals a goat. Afterwards, he asks you if you want to stick to your choice or if you like to change to another door.
2. Again, the host opens a door you have currently not chosen, making sure he is revealing a goat. Note, if you changed your choice in the previous round, the host might open the door you have initially chosen. After opening the door, he again asks if you want to stick to your choice or if you like to change to another door.

Finally, you get what is behind the door you are currently choosing. Assume the host to play fair and to always reveal a goat behind another door before one chooses.

Please use the following notation: For doors you have chosen, please use captial letters in alphabetic order:

- A_1 = you chose door A in round 1
- C_2 = you chose door C in round 2

For the doors opened by the host, please add a prime:

- A_1' = door A was opened by the host in round 1
- C_2' = door C was opened by the host in round 2

For the doors hiding the car, please use capital letters without indices or prime. Thus $p(C|A_1, B_1')$ is the probability that door C hides the car, give that you chose door A in round 1 and the host opened door B in round 1.

a) After the first round of the game, what are the probabilities for the remaining three doors to hide the car? Assume you chose door A and the host opened door B, thus in the above notation A_1, B_1'. Give a formal proof of your answer using Bayes' Theorem. ___/4

b) Assume you kept your choice in round 1. What are the probabilities for the remaining two doors to hide the car after round 2? Assume A_1, A_2, B_1', C_2'. Give a formal proof of your answer using Bayes' Theorem. ___/3

c) Assume you switched the door in round one. What are the probabilities for the remaining two doors to hide the car after round 2? Calculate the probabilities for the case the moderator opens the door you initially choose in round 1, and for the case, the host opened a door you have not chosen in round 1. For these two cases assume A_1, D_2, B_1', A_2' and A_1, D_2, B_1', C_2' respectively. Give a formal proof of your answer using Bayes' Theorem. ___/4

d) Based on your answers of b) and c), what is the optimal strategy to play, assuming you prefer cars over goats? What is your chance to win the car if you play the optimal strategy? ___/4

e) Now, you play an extended version of the game. There are 100 doors, 99 hide goats, and one a sportscar? The rules of the game are the same, but now there are 98 rounds in which the host opens a door and asks if you like to switch. Thus you play until only two doors are left. What is the optimal strategy to win the sportscar. If one plays the optimal strategy, what is the probability of winning? A formal proof is not needed, but explain your answers. ___/3

Question 4 ___/13

Consider the following coin toss experiment:

- A number n_A respectively n_B coin tosses are performed with coin A and coin B. The results are stored in data vectors $d_A^{(n)} = (d_1, \ldots, d_n) \in \{0,1\}^n$ and $d_B^{(n)}$, where 1 and 0 represent the possible outcomes head and tail, respectively.
- Individual tosses are independent from each other.
- All tosses are done with either coin A or coin B with each having an unknown bias $f_A, f_B \in [0,1]$ denoting the probability of head $P(\text{head}|f_i) = f_i \; \forall i \in \{A, B\}$.

Coin A and B are used in a game in which you are paid $G_A = 1$ € if you get a 1 with coin A and $G_B = 4$ € if you get a 1 with coin B. If the coin shows 0, you are not paid anything. You are now given the following data $d_A^{(2)} = (1,1)$, $d_B^{(2)} = (0,0)$.

Hint: For your convenience, we note that for $a, b \in \mathbb{N}$ it holds that $\int_0^1 dx\, x^a (1-x)^b = \frac{a!b!}{(a+b+1)!}$.

a) Derive the expected financial gain for each coin and decide which coin to toss next. ___/5

b) When is the optimal point in time to switch to the other coin assuming your chosen coin fails in subsequent tosses. ___/3

c) How much information did you gain on f_i for your chosen coin $i \in \{A, B\}$ by these additional (loosing) tosses in part b)? Please also state the units of the information gain in your answer. ___/5

Question 5 ___/20

You are informed that a three sided die with its sides having 0, 1, and 2 eyes shows on average $\frac{1}{2}$ eyes.

a) How much information (in nits, rounded to two digits) is provided thereby about the outcome of the next dicing? ___/8

Hint: Substitute $y := e^{\lambda}$ if λ is your Lagrange multiplier. In case you find multiple possible solution for y choose the one that gives a real λ.

b) Now you are informed that the average of $\frac{1}{2}$ eyes was actually obtained from only two tosses. How much information does this revised information provide on the next outcome of the die (with respect to having no information about the behavior of the die)? ___/12

Hint: First work out the knowledge on frequencies $f_i := P(\text{die shows } i|I)$ for $i \in \{0, 1, 2\}$ as encoded in $\mathcal{P}(f|I)$, where $f = (f_0, f_1, f_2)$ is the frequency vector. Use the uninformative prior $\mathcal{P}(f) = 2\,\delta\left(\left[\sum_{i=0}^{2} f_i\right] - 1\right)\left[\prod_{i=0}^{2} \theta(f_i)\,\theta(1-f_i)\right]$ with $\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ the Heaviside theta function. Then derive from this the probabilities of the outcome $p_i := \langle f_i \rangle_{(f|I)}$. The verification of the prior normalization might be useful for your calculation:

$$
\begin{aligned}
1 &\stackrel{?}{=} \int df_0 \int df_1 \int df_2 \mathcal{P}(f) \\
&= \int df_0 \int df_1 \int df_2\, 2\,\delta(f_0 + f_1 + f_2 - 1) \prod_{i=0}^{2} \theta(f_i)\,\theta(1-f_i) \\
&= \int_0^1 df_0 \int_0^{1-f_0} df_1 \int_0^{1-f_0-f_1} df_2 2\,\delta(f_0 + f_1 + f_2 - 1) \\
&= \int_0^1 df_0 \int_0^{1-f_0} df_1 2 \\
&= \int_0^1 df_0 2\,(1 - f_0) \\
&= 2\left(1 - \frac{1}{2}\right) = 1
\end{aligned}
$$

For your convenience, we note that for $a, b \in \mathbb{N}$ it holds that $\int_0^1 dx\, x^a (1-x)^b = \frac{a!b!}{(a+b+1)!}$.