Exam on Information Field Theory

Name: ____________________ Matriculation number: ______________

- The exam consists of **five exercises**. Please do check if you received all of them.
- In total one may gain 90 points. However it is likely to achieve the top grade with less points.
- The working time is **90 minutes**.
- **No aids** are allowed during the exam.
- In order to gain the full amount of possible points we strictly advise you to use conditional probabilities.

Question	**Points**
1	____/20
2	____/20
3	____/20
4	____/10
5	____/20
Bonus	
Total	____/90

Grade	

Recommendation: If you run into problems with one exercise, jump to the next one.
GOOD LUCK!

Question 1 ___/20

Mark whether the following statements are *always true* (T) or not (F).

a) [T] [F] Rational beliefs can be described as probabilities. ___/1

b) [T] [F] A statement and its negation have different truth values. ___/1

c) [T] [F] A statement and its negation have different probabilities. ___/1

d) [T] [F] A rational mind expects a *fair* coin which has shown head ten times in a row to show tail with the next toss with higher probability than head. ___/2

e) [T] [F] A rational mind expects a *dubious* coin which has shown head ten times in a row to show head with the next toss with higher probability than tail. ___/2

f) [T] [F] The Wiener filter is a linear operation on the data. ___/1

g) [T] [F] A statistically homogeneous field on $\mathbb{R}^u$ has a Fourier power spectrum $P(\vec{k})$ with $P(\vec{k}) = P(k)$, where $k = \left|\vec{k}\right|$. ___/1

h) Assume a signal $s \in \mathbb{R}$ and different estimates m for its value.

1) [T] [F] The posterior mean m minimizes $\langle |s-m| \rangle_{\mathcal{P}(s|d)}$. ___/1

2) [T] [F] The posterior median m minimizes $\langle |s-m| \rangle_{\mathcal{P}(s|d)}$. ___/1

3) [T] [F] The posterior median m minimizes $\langle -\delta\,(s-m) \rangle_{\mathcal{P}(s|d)}$. ___/1

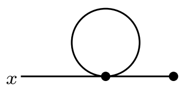
4) [T] [F] The maximum a posteriori m minimizes $\langle -\delta\,(s-m) \rangle_{\mathcal{P}(s|d)}$. ___/1

5) [T] [F] The maximum a posteriori m minimizes $\left\langle |s-m|^2 \right\rangle_{\mathcal{P}(s|d)}$. ___/1

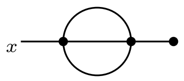
i) Assume the Hamiltonian

$$H(d,s) = -j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \lambda^\dagger s^4 \quad (1)$$

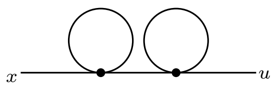
(remember that $\lambda^\dagger s^4 = \int \mathrm{d}x\ \lambda_x s_x^4$) and the convention that indices that appear more than once in one term are integrated over.

1) [T] [F] 

$= -\frac{1}{2} D_{xy} \lambda_y D_{yz} D_{zz} j_z$ ___/2

2) [T] [F] 

$= +\frac{1}{3!} D_{xy} \lambda_y \left(D_{yz}\right)^3 \lambda_z D_{zu} j_u$ ___/2

3) [T] [F] 

$= +\frac{1}{8} D_{xy} \lambda_y D_{yy} D_{yz} \lambda_z D_{zz} D_{zu}$ ___/2

Question 2 ___/A

signal of known shape r_x in u-dimensional position space ($x \in \mathbb{R}^u$) but completely unknown amplitude s ($s \in \mathbb{R}$ or $s \in \mathbb{C}$) is contained in noisy data d_x over the x-space. The additive noise is Gaussian with known covariance $N = \left\langle n n^\dagger \right\rangle_{(n)}$.

a) erive the matched filter, which is the signal mean given the above information (3 points). ___/D

b) ssume the noise covariance to be diagonal in position-space and write down the matched filter (2 points). ___/A

c) ssume the noise covariance to be diagonal in Fourier-space and write down Fourier-space version of the matched filter (1 point). ___/A

Question 3 ___/20

Consider a real-valued signal field s with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S), \tag{2}$$

that is observed with an instrument that exhibits an almost linear response,

$$d = R\left(s + rs^2\right) + n. \tag{3}$$

Here, R is a linear operator, $r \in \mathbb{R}$ with $|r| \ll 1$ is a small parameter that determines the strength of the nonlinearity in the instrumental response, s^2 denotes the local squaring of the signal field, i.e., $\left(s^2\right)_x = \left(s_x\right)^2$, and n is additive Gaussian noise, i.e.,

$$\mathcal{P}(n) = \mathcal{G}(n, N). \tag{4}$$

a) Consider first the case of an exactly linear response, i.e., $r = 0$. Derive the Hamiltonian ___/4

$$H(d, s) = -\log\left(\mathcal{P}(d, s)\right) \tag{5}$$

for this problem. You may drop all terms that do not depend on s.

b) Show that the posterior probability density in the case with $r = 0$ is of Gaussian form, i.e., $\mathcal{P}(s|d) = \mathcal{G}(s - m_0, D)$, and derive expressions for its mean and covariance, ___/4

$$m_0 = \langle s \rangle_{\mathcal{P}(s|d)} \qquad \text{and} \qquad D = \left\langle (s - m_0)(s - m_0)^\dagger \right\rangle_{\mathcal{P}(s|d)}, \tag{6}$$

as a function of d, S, N, and R.

c) Now consider the case with small but non-zero r. Calculate the Hamiltonian in this case and write it in the form ___/6

$$H(s, d) = H_0 - j^\dagger s + \frac{1}{2} s^\dagger D^{-1} s + \sum_{k=2}^{\infty} \frac{1}{k!} \Lambda^{(k)}_{x_1 x_2 \cdots x_k} s_{x_1} s_{x_2} \cdots s_{x_k}, \tag{7}$$

where only the coefficients $\Lambda^{(k)}$ depend on r and we use the convention that repeated indices are integrated over. Give expressions for j, D, and all non-zero $\Lambda^{(k)}$. You do not need to calculate H_0.

d) Write down the diagrammatic expansion of the partition function $\log\left(Z(d)\right)$ up to linear order in r. ___/3

e) Find the diagrammatic expressions for the posterior mean and covariance, ___/3

$$m_r = \langle s \rangle_{\mathcal{P}(s|d,r)} \qquad \text{and} \qquad \left\langle (s - m_r)(s - m_r)^\dagger \right\rangle_{\mathcal{P}(s|d,r)}, \tag{8}$$

up to first order in r.

Question 4 ___/20

Consider a damped harmonic oscillator that is exposed to external fluctuations. The oscillator's spatial displacement, $x = x(t)$, is a function of time as described by the following differential equation,

$$(\alpha\ddot{x} - \sigma\xi) + \dot{x} + \nu x = 0 \qquad \text{with} \qquad \alpha, \sigma, \nu \in \mathbb{R}. \tag{9}$$

Here, the external fluctuations, $\xi = \xi(t)$, are Gaussian white noise; i.e., $\xi \curvearrowleft \mathcal{G}(\xi, \Xi)$ with $\Xi = \Xi(t, t') = \delta(t - t')$.

a) Derive the Fourier representation $\Xi(\omega, \omega')$ of the autocorrelation function $\Xi(t, t')$ of the noise process that is driving the external fluctuations, and read off its Fourier power spectrum, $P_\xi(\omega)$. ___/2

b) Calculate the Fourier power spectrum, $P_x(\omega)$, of the spatial displacement. ___/5

<u>Hint</u>: Your result should be of the form,

$$P_x(\omega) \equiv \frac{A\sigma^2}{B^2 + (1 - C\alpha)\omega^2 + D\alpha\omega^4} \qquad \text{with} \qquad A, B, C, D \in \mathbb{R}, \tag{10}$$

which you are allowed to use in **c)** and **d)** instead of your own result if need be.

Assume a direct measurement of the displacement, $d = d(t)$, that is subject to additive noise, $n = n(t)$, of the following form,

$$d = x + n. \tag{11}$$

The noise is Gaussian white noise; i.e., $n \curvearrowleft \mathcal{G}(n, N)$ with $N(t, t') = \sigma^2\delta(t - t')$, where σ is identical to the one used in equations (9) and (10), since the noise is also caused by the external fluctuations.

c) Calculate the Fourier representation of the *a posteriori* mean, $m(\omega) = \langle x(\omega) \rangle_{(x|d)}$, and covariance, $D(\omega, \omega') = \langle [x(\omega) - m(\omega)][x(\omega') - m(\omega')]^* \rangle_{(x|d)}$, assuming a Wiener filter. ___/5

Let the external fluctuations dominate the displacement. In consequence, the inertia force of the oscillator becomes negligible; i.e., the system is sufficiently described by the limit $\alpha \to 0$.

d) Find the (complex) poles of the Fourier power spectrum calculated in **b)** in the case of dominating external fluctuations and calculate the corresponding autocorrelation function, $X(t, t') = \langle x(t)x(t') \rangle_{(x)}$. ___/8

Question 5 ___/10

The decay of a known amount m_0 of a radioactive isotope with unknown decay rate ν is measured. The amount left after the time $t \geq 0$ is

$$m_t = m_0 \mathrm{e}^{-\nu t}, \tag{12}$$

where ν is our signal. Assume a flat prior, $\mathcal{P}(\nu) = \text{const}$.

On average a fraction κ of the decays are registered in the time interval from $t = n\Delta$ to $t = (n+1)\,\Delta$ for all $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and stored in a data vector $d = (d_0, d_1, d_2, \dots)$.

Assume Poissonian statistics for the count rates with

$$\mathcal{P}(d_n|\lambda_n) = \frac{\lambda_n^{d_n} \mathrm{e}^{-\lambda_n}}{d_n!} \tag{13}$$

and the expected detection rate

$$\lambda_n(\nu) = \kappa \int_{n\Delta}^{(n+1)\Delta} \mathrm{d}t' \left(-\frac{\mathrm{d}}{\mathrm{d}t'} m_{t'} \right) = \kappa \left(m_{n\Delta} - m_{(n+1)\Delta} \right). \tag{14}$$

a) Give the Hamiltonian $H(d, \nu)$, disregarding any terms that are not ν-dependent. Hint: ___/4

$$\sum_{n=0}^{\infty} \mathrm{e}^{-n\Delta\nu} = \sum_{n=0}^{\infty} \left(\mathrm{e}^{-\Delta\nu}\right)^n = \frac{1}{1 - \mathrm{e}^{-\Delta\nu}} \tag{15}$$

b) Find the maximum a posteriori estimator for ν given d. ___/2

c) Derive the Gaussian approximation ___/4

$$\mathcal{P}(\nu|d) \approx \mathcal{G}(\nu - \nu_{\max}, D) \tag{16}$$

of the posterior around its maximum, i.e. give $\nu_{\max}$ and D using a saddle point approximation.

Question 6 ___/20

Consider a real-valued signal field s with a Gaussian prior,

$$\mathcal{P}(s) = \mathcal{G}(s, S),$$

which is observed with an instrument that exhibits a linear response,

$$d = R(s) + n = Rs + n.$$

Here, R is a linear operator and n is additive Gaussian noise, i.e.,

$$\mathcal{P}(n|s) = \mathcal{G}(n, N).$$

a) Derive the Hamiltonian ___/3

$$H_{\mathcal{G}}(d, s) = -\log\left(\mathcal{P}(d, s)\right)$$

for this problem and introduce the information propagator $D^{-1} = S^{-1} + R^{\dagger} N^{-1} R$ as well as the information source $j = R^{\dagger} N^{-1} d$. You may drop all terms that do not depend on s.

b) Calculate the moment generating functional (partition function), ___/2

$$Z_{\mathcal{G}}(J) = \int \mathcal{D}s \; e^{-H_{\mathcal{G}}(d,s) + J^{\dagger} s}. \tag{17}$$

c) Now suppose that the assumption of a linear response was wrong and we have to include an interaction term, ___/3

$$H_{\text{int}}(s) = \frac{\lambda}{3!} \int \mathrm{d}x \; s^3(x) + \mathcal{O}(s^4), \;\; \lambda \in \mathbb{R},$$

in the Hamiltonian (the $\mathcal{O}(s^4)$ terms ensure proper normalizability of the probability distribution function, but can be assumed to be negligible in this calculation). Thus, the correct Hamiltonian is given by

$$H(d, s) = H_{\mathcal{G}}(d, s) + H_{\text{int}}(s).$$

Show that the moment generating functional up to first order in λ can be written as

$$Z(J) = \left\{ 1 - H_{\text{int}}\left(\frac{\delta}{\delta J}\right) + \dots \right\} Z_{\mathcal{G}}(J). \tag{18}$$

d) Determine the generating functional, which is given by equation (17) and (18), up to first order in λ, either by a calculation or by Feynman diagrams. You should finally obtain ___/6

$$\begin{aligned} Z(J) = \exp\Bigg\{ &-\frac{\lambda}{2} \int \mathrm{d}x \int \mathrm{d}y \; D(x,x) D(x,y) (J+j)(y) \\ &-\frac{\lambda}{6} \left(\int \mathrm{d}x \int \mathrm{d}y \; D(x,y)(J+j)(y) \right)^3 \Bigg\} Z_{\mathcal{G}}(J), \end{aligned} \tag{19}$$

after re-identifying $1 - x \approx e^{-x}$.

e) Finally calculate the mean and covariance of s by the use of equation (19) and rewrite the analytical expressions in terms of *Feynman* diagrams. ___/6