Simulating the Universe: Methods and Applications

Magnetohydrodynamics & Accretion Disks

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- MHD Equations
- MHD Effects
- Simulations and Results

Magnetic fields in the universe

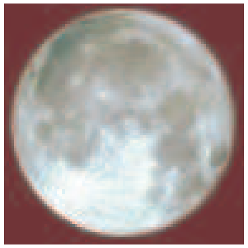
Sun

up to $3 \cdot 10^{3}$ G

Earth

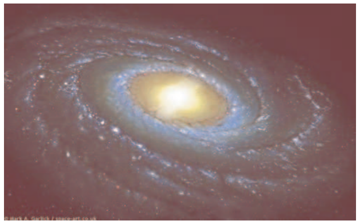
$3 \cdot 10^{-1}$ G

Moon



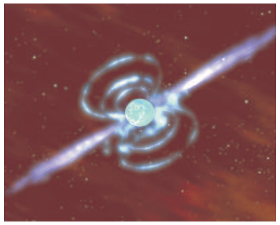
no magnetic field

Milkyway



10^{-6} G

Neutron star



up to 10^{15} G

Hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0$$ continuity equation

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \operatorname{div}\left[\rho(\vec{v} \otimes \vec{v})\right] + \operatorname{div}\Pi = -\rho \operatorname{grad}\phi$$ momentum equation

$$\frac{\partial}{\partial t}(\rho E) + \operatorname{div}\left[(\rho E + p)\vec{v}\right] + \operatorname{div}\vec{h} - \operatorname{div}(\vec{\vec{\pi}}\,\vec{v}) = -\rho \vec{v} \operatorname{grad}\phi$$ energy equation

$$\Pi = p \cdot \mathrm{I} - \vec{\vec{\pi}}$$

p: isotropic gas pressure

h: heat transport $\quad E = \frac{1}{2}\left|\vec{v}\right|^2 + \varepsilon$

$+$ closure relation

Assumptions for MHD

- all velocities much smaller than the speed of light
- no charges in the restframe of the fluid (charge neutrality)
- no electric fields in the restframe of the fluid
- isotropic pressure ($\boldsymbol{\pi} = 0$)
- no heat flow $(\vec{h} = 0)$

Maxwell equations

$\mathrm{div}\,\vec{E} = 4\pi\rho$ (Gauss‘ law) $\mathrm{MHD}: \rho = 0$

$\mathrm{div}\,\vec{B} = 0$ (no mag. monopoles)

$\mathrm{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ (Faraday‘s law of induction)

$rot\,\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ (Ampere‘s law) $\mathrm{MHD}: \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0$

$\eta\vec{j} = \vec{E} + \frac{v}{c}\vec{B}$ Ohm`s law

η : Resistivity

resistive MHD equations

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho \vec{v}\right) = 0$$

continuity equation

$$\frac{\partial}{\partial t}\left(\rho \vec{v}\right) + \mathrm{div}\left[\rho\left(\vec{v} \otimes \vec{v}\right)\right] + \mathrm{div}\, p_{tot} - \vec{j} \times \vec{B} = -\rho\, \mathrm{grad}\phi$$

momentum equation

$$\frac{\partial}{\partial t}\left(\rho E_{tot}\right) + \mathrm{div}\left[\left(\rho E_{tot} + p\right)\vec{v}\right] - (\gamma - 1)\eta\left|\vec{j}\right|^2 = -\rho \vec{v}\, \mathrm{grad}\phi$$

energy equation

+ Maxwell equations

$$p_{tot} = p + \frac{B^2}{8\pi} \qquad E_{tot} = E + E_{mag}$$

(we used the EOS of an ideal gas)

magnetic Reynolds number: $R_m = \frac{v_0\, l_0}{\eta}$

ideal MHD equations

$$\eta = \sigma^{-1} = 0$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\left(\rho \vec{v}\right) = 0$$

continuity equation

$$\frac{\partial}{\partial t}\left(\rho \vec{v}\right) + \mathrm{div}\left[\rho\left(\vec{v} \otimes \vec{v}\right)\right] + \mathrm{div}\, p - \mu_0^{-1}\left(\nabla \times \vec{B}\right) \times \vec{B} = -\rho\, \mathrm{grad}\phi$$

momentum equation

$$\frac{\partial}{\partial t}\left(\rho E\right) + \mathrm{div}\left[\left(\rho E + p\right)\vec{v}\right] = -\rho \vec{v}\, \mathrm{grad}\phi$$

energy equation

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{B}\right) = 0$$

$$\nabla \vec{B} = 0$$

=> coupled nonlinear partial differential equations

MHD in conservation form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\pi} = 0 \qquad \vec{\pi} = \rho \vec{v} \qquad M = \int \rho \, dV$$ mass conservation

$$\frac{\partial \vec{\pi}}{\partial t} + \nabla \cdot \mathrm{T} = 0 \qquad \mathrm{T} = \rho \vec{v}\,\vec{v} + \left(p + \tfrac{1}{2} B^2\right)\mathrm{I} - \vec{B}\,\vec{B} \qquad \Pi = \int \pi \, dV$$ momentum conservation

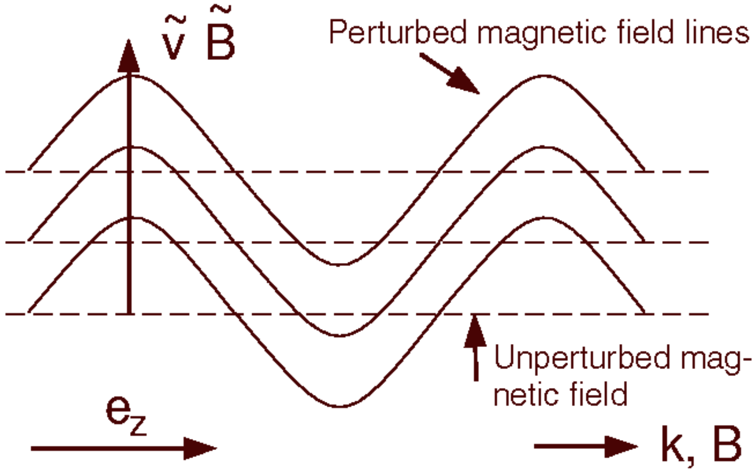
$$\frac{\partial H}{\partial t} + \nabla \cdot U = 0 \qquad H = \tfrac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \tfrac{1}{2} B^2 \qquad \mathrm{H} = \int H \, dV$$ energy conservation

$$U = \left(\tfrac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1} p\right)\vec{v} + B^2 \vec{v} - \left(\vec{v} \cdot \vec{B}\right)\vec{B}$$

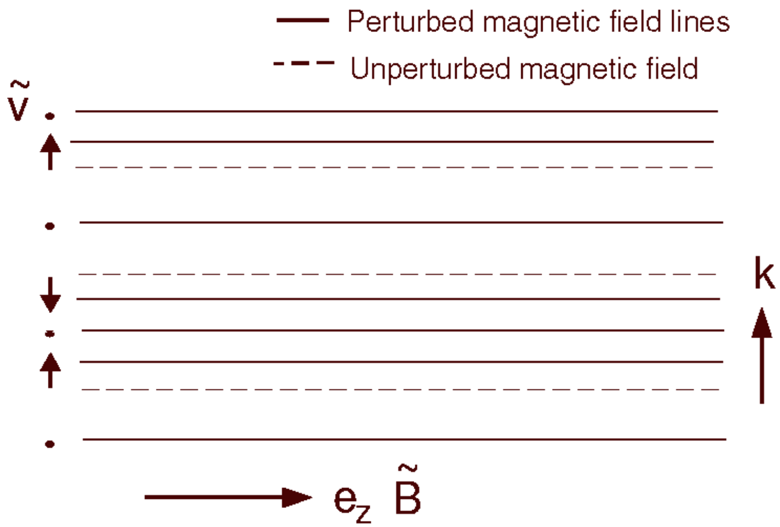
$$\frac{\partial \vec{B}}{\partial t} + \nabla \cdot \mathrm{Y} = 0 \qquad \mathrm{Y} = \vec{v}\vec{B} - \vec{B}\vec{v} \qquad \Psi = \int \vec{B} \cdot \vec{n} \, d\vec{\sigma}$$ flux conservation

primitive variables: $\rho, \vec{v}, p, \vec{B}$

Waves



transversal Alven waves



magnetosonic waves

Magnetic Dynamos

Diffusion equation

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B} + \nabla \times \left(\vec{v} \times \vec{B} \right) \qquad \eta = \frac{c^2}{4\pi\sigma} \quad \text{(diffusion coefficient)}$$

decay timescale (without source term): $\tau = \frac{l^2}{\eta} \approx 10^4$ years (earth)

=> some mechanism is needed to generate magnetic fiels => dynamo

Cowling‘s “Antidynamo“ Theorem

It is impossible for an axisymmetric configuration to maintain a steady-state magnetic field.

Accretion disks

Problem: how do fluid elements loose their specific angular momentum

Idea: viscosity, but viscosity cannot do this for large disks

(timescale $\sim 3 \cdot 10^7$a for $l \sim 10^{10}$cm $\rightarrow$ too long)

Solution: viscosity is enhanced by turbulence

New Problem: rotating Keplerian disks (rotation profile: $\omega \sim r^{-\frac{3}{2}}$) are stabilized by Coriolis forces even at large scales

Solution: magnetized disks may be able to create turbulence

Magnetorotational instability

Idea: a magnetic field destabilizes a differentially rotating fluid

considering a fluid element that is displaced in the orbital plane by $\vec{\xi}$ with spatial dependence e^{ikZ} one gets

$$\delta\vec{B} = ikB\vec{\xi}$$

and a magnetic tension force

$$\frac{ikB}{4\pi\rho}\delta\vec{B} = -\left(\vec{k}\cdot\vec{v}_A\right)^2\vec{\xi}$$

Stability criterium for MRI:

$$\frac{d\Omega^2}{dR} \geq 0$$

This instability also holds in the limit $B \rightarrow 0$. Therefore one should think of an accretion disk as a magnetized fluid with rotation instead of a rotating fluid with a magnetic field.

Numerical Simulations

Local approximation (shearing box)

focus lies on a small region of the disk

Limit:

$$R \to \infty \qquad v_\phi \to \infty \qquad \frac{v_\phi}{R} = \Omega \to \text{finite}$$

All velocities are measured relative to $R\Omega_0$

Boundary conditions:

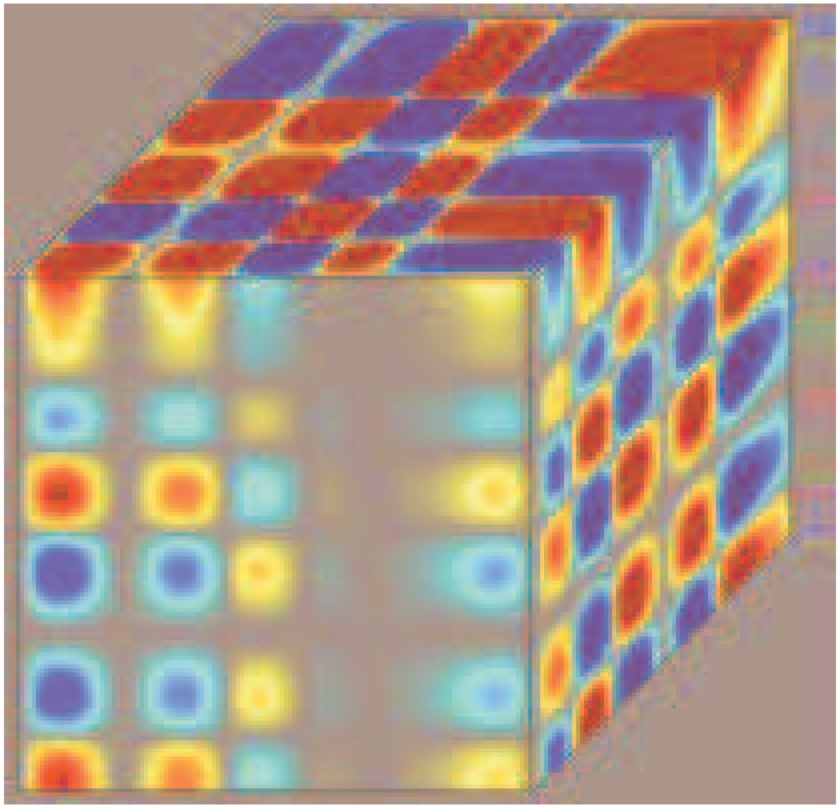
- azimuthal: periodic
- vertical: periodic or outflowing
- radial: quasi-periodic

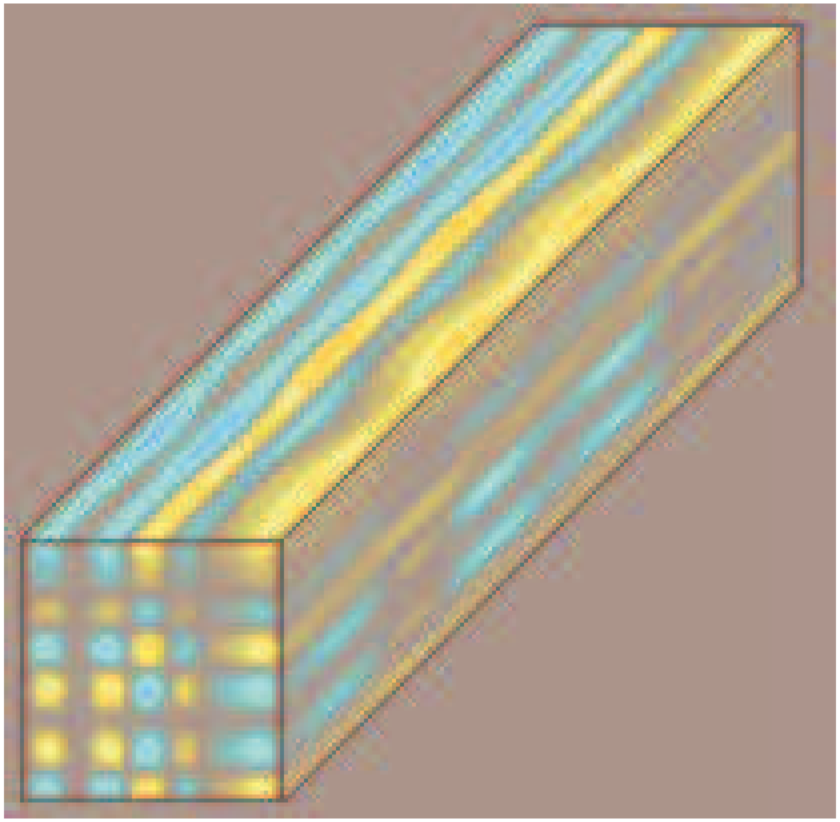
2D (axisymmetric)

- initial uniform field: nonlinearities not turbulent
- initial vertical field with vanishing mean value: turbulence develops soon, but decays leaving no field (antidynamo theorem)

3D

- homogeneous shearing box, containing only the radial component of the gravitational field
- initial vertical field + toroidal field with zero mean value
- a sufficiently large box develops instabilities, that lead to tubulence and enhanced angular momentum transport





Global simulations

- simulate the whole disk

- accreting Black hole: length scale ~ 10^3 r_s (r_s = Schwarzschild radius)
  factor $\sim 3 \cdot 10^4$ in timescales because of Keplarian rotation

- useful solution is a big computational problem, so
  one has to make compromises

global 2D simulations

- neglect z-coordinate
- Starting accretion from constant angular momentum torus $(\omega \sim r^{-2})$ with weak embedded poloidal field
- Result: barotropic disk and a magnetized coronal outflow enveloping the disk are created
- but: turbulence disappears (antidynamo again)

global cylindrical simulations

- full radial, azimuthal and vertical dynamics, but no vertical component of the gravitational field
- its possible to restrict the azimuthal range to some fraction of 2π without affecting the qualitative result of the simulation
- turbulence is developed rapidly
- Result: additional dynamo amplification of the magnetic field by MHD effects is reported

global 3D simulations

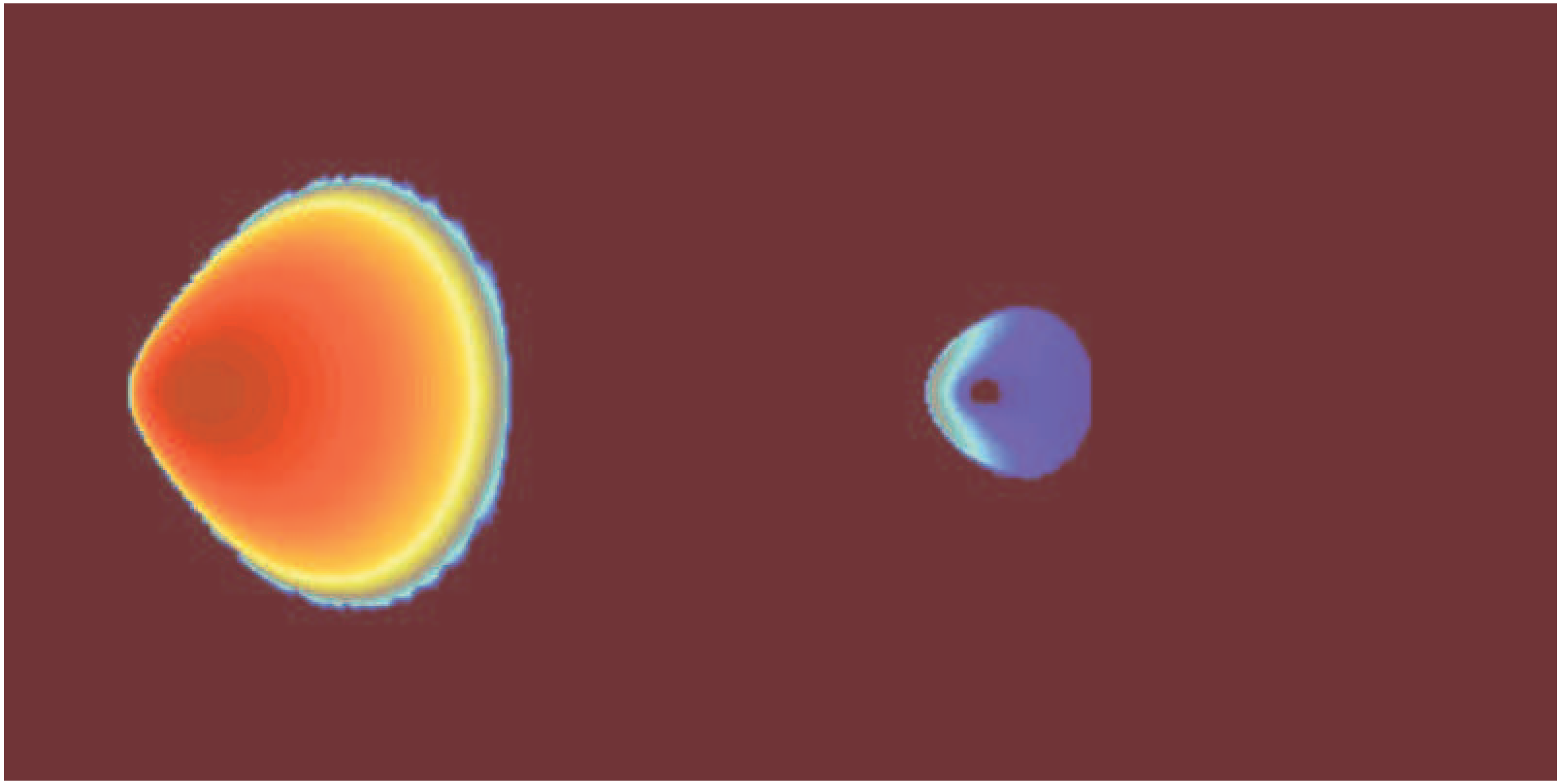
- initial: constant angular momentum torus with embedded magnetic field (toroidal and/or poloidal)

- initial angular momentum profile is changed to nearly Keplerian very fast (within a few orbital times at pressure maximum of the torus)

- instabilities are efficient enough to eliminate the initial pressure gradient

- significant angular momentum transport takes place

- 3 dynamical structures are created:
  - a hot, thin slightly sub-Keplarian disk
  - a strongly magnetized corona surrounding the disk
  - a jet-like outflow confined by magnetic coronal pressure



Conclusion of the numerical simulations

- magnetic fields turn gradient free energy sources (angular velocity or temperature) into sources of dynamical instability

- As long as the MHD approximations hold, magnetic fields maintain their influence even in the formal limit B = 0.

- Combination of a magnetic field and an outwardly decreasing differential rotation leads to the magnetorotational instability

- MRI leads to turbulence, that helps transporting energy and angular momentum outward

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