Special Relativistic Hydrodynamics and Relativistic Jets

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Outline

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Hydrodynamic Approximation

The matter of astrophysical objects can be approximated as fluid if:

- microscopic behaviour of single particle can be neglected:

$$\lambda \ll L$$

$\Rightarrow$ fluid elements and mean values can be defined

- "short range" or saturating forces between particles:

$$\lim_{N \to \infty} \left(\frac{E}{N} \right) = \text{const.}$$

$\Rightarrow$ definition of energy density and pressure possible

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Approximation applicable?

Gravity and electromagnetic forces are of long range:

- gravity must be included in the HD equations as external force
- electromagnetic forces saturate in electrical neutral systems due to screening, e. g. in plasmas

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Astrophysical Scenarios where RHD is at Work

Relativistic description necessary?

- velocity $\sim$ speed of light   or
- internal energy density $\sim$ rest-mass density   or
- gravitational fields "strong"

Examples:

- Active galactic nuclei (AGNs)
- Microquasars
- Gamma ray bursts (GRBs)
- Pulsar Winds

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Outline

The Equations of General Relativistic Hydrodynamics

Equations are local conservation laws:

$$J^{\mu}_{;\mu} = 0, \quad \text{conservation of mass}$$
$$T^{\mu\nu}_{;\mu} = 0, \quad \text{conservation of energy-momentum}$$

Covariant derivative:

$$A^{\nu}{}_{;\mu} = A^{\nu}{}_{,\mu} + \Gamma^{\nu}_{\sigma\mu} A^{\sigma}; \qquad \Gamma^{\nu}_{\sigma\mu} = \frac{1}{2} g^{\nu\lambda} \left(g_{\lambda\mu,\sigma} + g_{\lambda\sigma,\mu} - g_{\sigma\mu,\lambda} \right)$$

Perfect fluid (no shear or heat conduction) ($G = c = 1$):

$$J^{\mu} = \rho u^{\mu}, \quad T^{\mu\nu} = \rho h u^{\mu} u^{\nu} + p g^{\mu\nu}$$

Equation of state (EoS) closes the system: $p = p(\rho, \epsilon)$

Approximations

Special Relativity (RHD)	$g_{\mu\nu} = \eta_{\mu\nu}$ gravitational field neglected	- Rel. heavy-ion collisions - Extragalactic jets - GRB afterglows
External Field (GRHD)	$g_{\mu\nu} \neq f(t)$ backgr. metric	- Accr. on comp. objects - jet formation - proto-GRBs
Dynamical $\mathcal{M}$ (GRHD + EE)	$g_{\mu\nu} = f(t)$ $g_{\mu\nu}$ from EE	- GR stellar core collapse - NS, BH mergers

The Equations of Special Relativistic Hydrodynamics

$$g^{\mu\nu} = \text{const.} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

$$\Rightarrow \Gamma^{\nu}_{\sigma\mu} = 0 \Rightarrow A^{\nu}{}_{;\mu} = A^{\nu}{}_{,\mu} \Rightarrow$$

$$J^{\mu}_{,\mu} = 0$$

$$T^{\mu\nu}_{,\mu} = 0$$

The Equations of SRHD in Conservation Form

Use $u^\mu = \Gamma(1, v^i)$ with $u^\mu u_\mu = -1$, $\Gamma = \frac{1}{\sqrt{1-|\mathbf{v}|^2}}$:

RHD Equations in Conservation Form

$$\mathbf{U}_{,t} + \mathbf{F}^{(i)}{}_{,i} = 0 \tag{1}$$

With $\mathbf{U}(\mathbf{W}) \equiv (D, S^j, \tau)$: conserved variables (unknowns),
$\mathbf{W} \equiv (\rho, v^i, \epsilon)$: primitive variables,
$\mathbf{F}^{(i)}(\mathbf{W}) = (Dv^i, S^j v^i + p\delta^{ji}, S^i - Dv^i)$: vectors of fluxes,

and $D = \rho\Gamma \equiv$ rest-mass density,
$S^j = \rho h \Gamma^2 v^j \equiv$ momentum density,
$\tau = \rho h \Gamma^2 - p - \rho\Gamma \equiv$ energy density.

Comparison with Non-relativistic Limit

Equations in vector notation:

$$\begin{pmatrix} D \\ \mathbf{S} \\ \tau \end{pmatrix}_{,t} + \begin{pmatrix} Dv^i \\ \mathbf{S}v^i + \mathbf{e}_j p\delta^{ij} \\ S^i - Dv^i \end{pmatrix}_{,i} = 0$$

1D, non-relativistic Equations:

$$\begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix}_{,t} + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (\rho E + p)v \end{pmatrix}_{,x} = 0$$

Hyperbolicity, Characteristics, RH Jump Conditions

Hyperbolicity:

- Given for (1) if EoS causal: $c_s < 1$.
- Characteristics (1D): $\frac{dx}{dt} = \lambda_k(\mathbf{U}(x,t))$

  3D: $\lambda_0 = v^i$,
  $$\lambda_\pm = \frac{1}{1-v^2c_s^2}\left(v^i(1-c_s^2) \pm c_s\sqrt{(1-v^2)\left[1-v^2c_s^2 - v^iv^i(1-c_s^2)\right]}\right)$$
  1D: $$\lambda_\pm = \frac{v\pm c_s}{1+vc_s} \rightarrow \begin{cases} 1 & (v\rightarrow 1) \\ v\pm c_s & (v, c_s \rightarrow 0) \end{cases}$$

- Important: Hyperbolic equations admit discontinuous solutions (shocks), but fluxes have to be continuous across shocks $\rightarrow$ Rankine-Hugoniot jump conditions:

$$\begin{aligned} [\rho u^\mu]n_\mu &= 0 \\ [T^{\mu\nu}]n_\mu &= 0 \end{aligned} \quad \xrightarrow{non-rel.} \quad \begin{aligned} D[\rho] - [\rho v] &= 0 \\ D[\rho v] - [\rho v^2 + p] &= 0 \\ D[\rho E] - [v(\rho E + p)] &= 0 \end{aligned}$$

Why is RHD more Difficult than Non-relativistic HD?

What is different from classical HD?

- Equations are tightly coupled by $\Gamma = \Gamma(v^x, v^y, v^z)$ and $h = h(\rho, \epsilon)$. (5 equations for the 5 unknowns $\mathbf{W} = (\rho, \mathbf{v}, \epsilon)$)
- No explicit relation between **W** and **U** (except for particular EoS).
- Coupling of the tangential components of v in the characteristic speeds $\lambda_\pm$ (aberration).
- For $v \rightarrow 1$ the characteristic speeds degenerate to $c = 1 \Rightarrow$ very thin structures (e. g. rel. blast waves) $\Rightarrow$ potential source of numerical errors.
- Relativistic shocks can have larger jumps than classical ones:

$$\left(\frac{\rho_b}{\rho_a}\right)_{rel.} \leq \frac{\gamma\Gamma_b + 1}{\gamma - 1} \xrightarrow{v_b \rightarrow 1} \infty \leftrightarrow \left(\frac{\rho_b}{\rho_a}\right)_{cl.} \leq \frac{\gamma + 1}{\gamma - 1} \sim 4 - 7$$

Outline

Numerical Methods, Artificial Viscosity

General Possibilities:

- Finite difference/ volume techniques: treated in this talk
- Smooth particle hydrodynamics
- Spectral techniques

Artificial Viscosity (Used in the late 60's - 80's):

- Standard finite difference techniques used to discretize differential RHD equations
- Viscous terms added to dump spurious oscillations near discontinuities
  $\longrightarrow$ artificial dissipative mechanism $\Rightarrow$ shock transition smoothed, extended over several zones

Artificial Viscosity

Limitations of artificial viscosity:

- very diffusive
- errors in shock velocity
- test dependent parameters
- non conservative
- only applicable for $\Gamma \leq 2$

Applications:

- axisymmetric stellar collapse
- accretion onto compact objects
- numerical cosmology

High Resolution Shock Capturing (HRSC) Methods

Used since the 90's, caused a "revolution" in numerical RHD because:

- The equations are written in conservation form.
  $\Rightarrow$ Convergence to the physically correct solution
- They exploit the hyperbolic character of the RHD equations.
  $\Rightarrow$ RH jump conditions automatically satisfied ("shock capturing")
- "High resolution":
  $\Rightarrow$ High order of accuracy in smooth regions of the flow
  $\Rightarrow$ Stable and sharp description of discontinuities

Basics of HRSC Methods 1

We want our schemes to conserve quantities.
$\Rightarrow$ Use finite volume schemes especially when the solution involves shock waves.

- 1D example: general continuity equation:

$$\frac{\partial}{\partial t}U(x,t) = -\frac{\partial}{\partial x}F(U(x,t)), \quad \text{e. g. } \rho_t = -(\rho v)_x$$

- introduce cells of width Δx and Δt:

$$\overset{\int_{x_1}^{x_2} dx}{\Longrightarrow} \frac{\partial}{\partial t}\int_{x_1}^{x_2} U(x,t)\,dx = F(U(x_1,t)) - F(U(x_2,t))$$

$$\overset{\int_{t_1}^{t_2} dt}{\Longrightarrow} \int_{x_1}^{x_2} U(x,t_2)\,dx = \int_{x_1}^{x_2} U(x,t_1)\,dx + \int_{t_1}^{t_2} F(U(x_1,t))\,dt - \int_{t_1}^{t_2} F(U(x_2,t))\,dt$$

Basics of HRSC Methods 2

- Divide byΔx and introduce cell averaged state vectors $\hat{U}_j^n$ and time averaged fluxes $\hat{F}_j^n$:

$$\underbrace{\frac{1}{\Delta x}\int_{x_1}^{x_2} U(x,t_2)\,dx}_{\hat{U}_1^2} = \underbrace{\frac{1}{\Delta x}\int_{x_1}^{x_2} U(x,t_1)\,dx}_{\hat{U}_1^1} -$$

$$\frac{\Delta t}{\Delta x}\left[\underbrace{\frac{1}{\Delta t}\int_{t_1}^{t_2} F(U(x_2,t))\,dt}_{\hat{F}_2^1} - \underbrace{\frac{1}{\Delta t}\int_{t_1}^{t_2} F(U(x_1,t))\,dt}_{\hat{F}_1^1}\right]$$

- In general we have: $\hat{U}_i^{n+1} = \hat{U}_i^n - \frac{\Delta t}{\Delta x}\left(\hat{F}_{i+1}^n - \hat{F}_i^n\right)$.

Basics of HRSC Methods 3

$$\hat{U}_i^{n+1} = \hat{U}_i^n - \frac{\Delta t}{\Delta x}\left(\hat{F}_{i+1}^n - \hat{F}_i^n\right) \tag{2}$$

$$\hat{F}_i^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} F(U(x_i, t))\, dt$$

$\Rightarrow$ To compute the fluxes, we need to know the solution $U(x_i, t)$ and $U(x_{i+1}, t)$ for t between t_n and t_{n+1}.

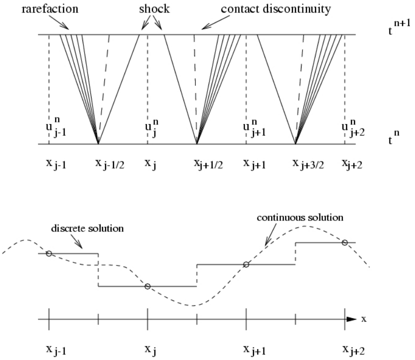
$\Rightarrow$ Conservation:

$$\sum_{i=1}^{N} \hat{U}_i^{n+1}\Delta x = \sum_{i=1}^{N} \hat{U}_i^n \Delta x - \Delta t\left(\hat{F}_{N+1}^n - \hat{F}_1^n\right)$$

Basics of HRSC Methods 4

- Idea: $U(x_i, t)$ can be calculated by solving Riemann problems at every zone interface.
- Known: U_i^n for all i $\Rightarrow$ construct piecewise constant function with steps at zone interfaces

  $\Rightarrow$ solve a series of Riemann problems with appropriate algorithm ("Riemann solver")



Basics of HRSC Methods 5

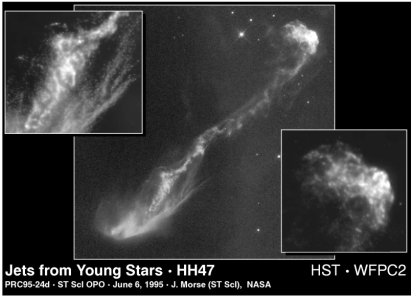
High Resolution?

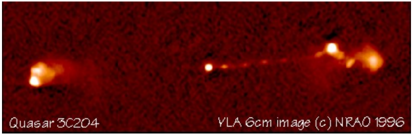
- If piecewise constant states are used (Godunov's method) only first order accuracy is achieved.
  $\longrightarrow$ Second order: piecewise linear functions (MUSCL, ...)
  $\longrightarrow$ Third order: piecewise quadratic functions (PPM, PHM, ...)
- There exist HRSC methods which are not Riemann solver based, and also non HRSC methods but they are not treated here.

General Properties of Jets

Observable Facts:

- highly collimated gas streaming out on two sides of a compact central source often at relativistic velocities
- length scales: from stellar to Mpc scales
- very strong radio emitters: "radio lobes" at both ends
- often: x-ray or radio knots along the arms





Jets are found in YSOs, Microquasars, AGNs, GRBs, ...

Observable Facts

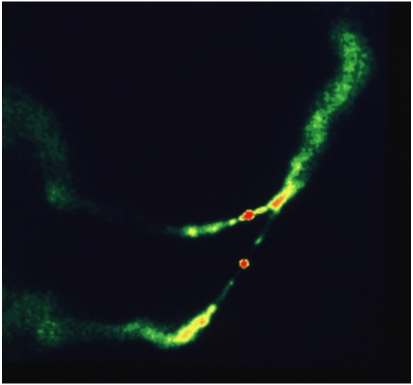


Image courtesy of NRAO/AUI and F.N. Owen, C.P. O'Dea, M. Inoue, J. Eilek

Twin Wide Angle Tail Radio Galaxy

Image courtesy of NRAO/AUI and Dana Berry, SkyWorks Digital

Artists conception of a GRB

Motivation of Simulations

The only source of information for Astronomical observers is the radiation emitted from some regions of these objects.

$\Rightarrow$ non (currently) observable properties of jets:

- morphology of the non-emitting regions (thermal plasma)
- jet composition
- jet creation

$\Rightarrow$ simulations are needed

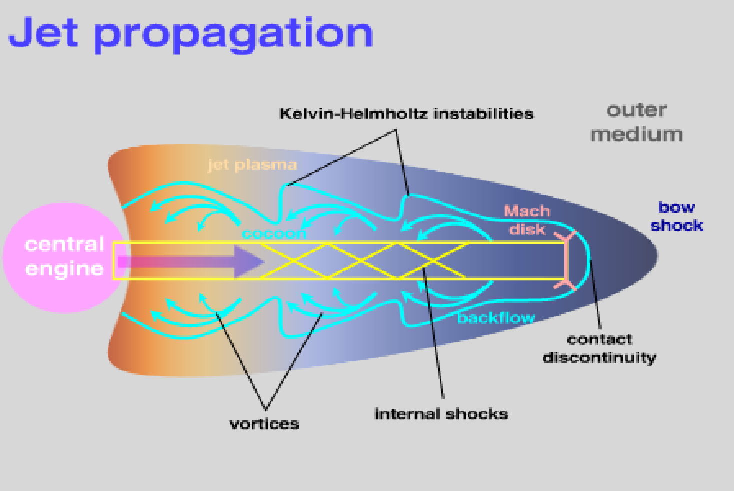
Morphology of Jets

Multidimensional RHD simulations possible since about 15 years have shown that relativistic jets display similar morphological properties as Newtonian jets:

- supersonic beam, almost constant diameter, collimation due to internal oblique shock waves and plane and centered rarefaction waves
- beam ends in a contact discontinuity — the working surface — separating ambient gas shocked by a leading bow shock from beam gas decelerated to the speed of the contact discontinuity by a trailing terminal shock configuration called Mach disk
- at beam cap: shocked, high preassure gas deflected sideways $\longrightarrow$ flows backwards along the beam:

  $\longrightarrow$ cocoon: supersonic, contains vortices, Kelvin-Helmholtz instabilities at boundaries which can cause turbulence

Morphology of Jets

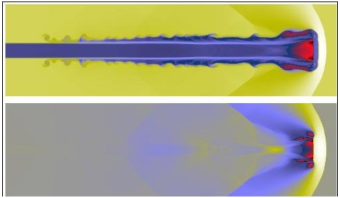


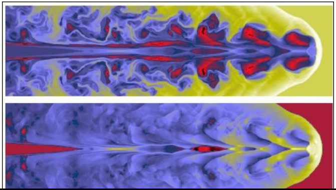
http://www.mpe.mpg.de/˜amueller/lexdt.html

The Morphology of Relativistic Jets

Relativistic effects: important differences between hot ($h \gg 1$) and cold ($h \approx 1$) relativistic jets:

Martí et al. 1997





- Hot jets have very thin cocoons (nearly no backflow) that surround a beam with nearly no inner structure. The reason is that the beam and its cocoon are in pressure equilibrium.
- Cold jets have thick turbulent cocoons (big backflow) surrounding a beam with internal shocks which are caused by the pressure gradient between beam and cocoon and by vortices in the cocoon.

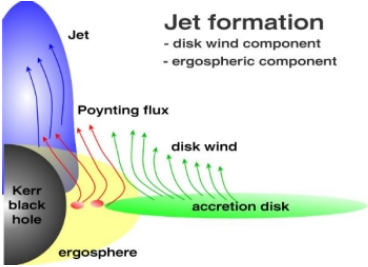
Jet Formation

Jets around rotating compact massive objects:

- gravitational energy set free in accretion disk is used to accelerate some of the matter of the disk
- acceleration not yet understood in detail but must involve hydromagnetic processes (Blandford & Payne (1982))

Jets around rotating black holes:

- Blandford-Znajek (1977): electrodynamic jet generation/acceleration



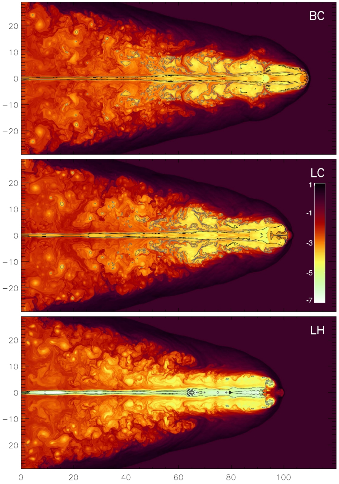
http://www.mpe.mpg.de/~amueller/lexdt.html

Jet Composition

Jets could be:

- leptonic
- baryonic
- electro-magnetic

Up to now no decision upon simulation data possible.



Simulation by Scheck et al. 2002

Summary

- Simulations are necessary along with theory and observations to unveil the nature of relativistic jets.
- Our advances in RHD have been largely triggered by the advance of numerical techniques (RHD more complex than Newtonian HD!!).
- Most successful techniques make explicit use of the conservation properties of the RHD system of equations.
- Outlook
    - Sufficiently large dynamical range to capture properly processes of mixing and mass entrainment (resolve turbulent scales!).
    - Proper coupling between thermal (hydrodynamics) and non-thermal plasma (emitting particles).
    - Simulations should guide our theoretical intuition.

Literature

- R. J. LeVeque, D. Mihalas, E. A. Dori, E. Müller, Computational Methods for Astrophysical Fluid Flow
- E. Müller, Vorlesung Hydrodynamik: Grundlagen und Numerische Verfahren (TU München, SS 2002)
- A. W. Guthmann et al.: Relativistic Flows in Astrophysics