Smoothed Particle Hydrodynamics (SPH) & Galaxies

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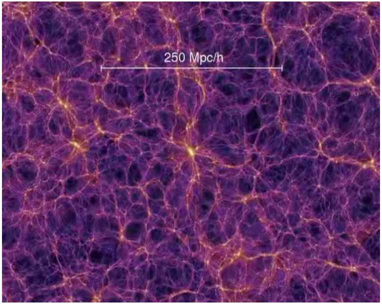
Outline

The End is the Beginning...

To remember:

- The gravitation in the universe is dominated by the dark matter
- The dynamics of the dark matter can be simulated by *PM* or P^3M tree methods
- This gives us the nice pictures from the last talk:



Problem Definition

As the visible universe also consists of baryonic matter, we have to simulate it too!
Baryonic matter is highly dissipative and can be treated as a hydrodynamic flow.
We need a numerical scheme that:

- Solves the hydrodynamic equations
- Is able to treat the dissipation correctly
- Is easy to implement in our existing tree algorithms

$\Rightarrow$ SPH (Smooth Particle Hydrodynamics) fullfills all points and :

- Is a free Lagrange method
- Does not need a grid to compute derivatives
- Theoretical results can easily be interpreted

Basic Equations

SPH is a numerical method for solving the hydrodynamic equations:

- The momentum equation :

$$\frac{\partial}{\partial t}\left(\rho\vec{v}\right) + div\left[\rho\left(\vec{v}\otimes\vec{v}\right)\right] + grad\ p \quad = \quad -\rho\ grad\ \phi$$

- The continuity equation :

$$\frac{\partial\rho}{\partial t} + div\left(\rho\vec{v}\right) = 0$$

- The Energy equation:

$$\frac{\partial}{\partial t}\left(\rho u\right) + div\left[\left(\rho u + p\right)\vec{v}\right] = 0$$

Smoothed Particle Hydrodynamics

Interpolation Function

One can interpolate a given function $A(\vec{r})$ by :

$$A_1(\vec{r}) = \int A(\vec{r}\,')\delta(\vec{r} - \vec{r}\,')d\vec{r}\,' \quad \int A(\vec{r}\,')W(\vec{r} - \vec{r}\,', h)d\vec{r}\,'$$

$W(\vec{r} - \vec{r}\,', h)$ is called the Kernel and h the Smoothing Length
The properties of the Kernel are:

- Its a normalized function: $\int W(\vec{r} - \vec{r}\,', h)d\vec{r}\,' = 1$
- It mimics delta function: $\lim_{h \to 0} W(\vec{r} - \vec{r}\,', h) = \delta(\vec{r} - \vec{r}\,')$

For computation we discretise the interpolant in mass elements $\rho(\vec{r}\,')d\vec{r}\,'$:

$$A_S(\vec{r}) = \int \frac{A(\vec{r}\,')}{\rho(\vec{r}\,')} W(\vec{r} - \vec{r}\,', h)\rho(\vec{r}\,')d\vec{r}\,' \propto \sum_b m_b \frac{A_b}{\rho_b} W(\vec{r} - \vec{r}\,', h)$$

$\Rightarrow$ In SPH any quantity is calculated this way !!

Smoothed Particle Hydrodynamics
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Smoothed Particle Hydrodynamics

How do we profit from this ?

The mass elements are randomly distributed over the simulated fluid
This interpolation is useful, because the hydrodynamic equations become a set of ordinary differential equations. For example :

$$\nabla A(\vec{r}) = \sum_b m_b \frac{A_b}{\rho_b} \nabla W(\vec{r} - \vec{r}', h)$$

Only the kernel has to be differentiated. As A_b is a fixed number for each particle.
For better accuracy one often symmetrizes the differentiation (ρ : density):

$$\rho \nabla A = \nabla(\rho A) - A \nabla \rho$$

This is useful to get conservation laws from this highly dissipative method.

Smoothed Particle Hydrodynamics
Kernels

Different kernels in SPH correspond to different difference schemes in grid methods. For physical Interpretation we use a gaussian kernel :

$$W(\vec{r}-\vec{r}', h) \propto e^{-\left(\frac{(\vec{r}-\vec{r}')^2}{h^2}\right)}$$

For numerics many other kinds of kernels are used, for example one based on splines

$$W(\vec{r}, h) = \frac{\sigma}{h^\nu} \begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & if 0 \leq \frac{r}{h} \leq 1; \\ \frac{1}{4}(2-q)^3 & if 1 \leq \frac{r}{h} \leq 2; \\ 0 & otherwise \end{cases}$$

with ν the number of dimensions, $q = \vec{r} - \vec{r}'$ and $\sigma \epsilon \left[\frac{2}{3}, \frac{10}{7\pi}, \frac{1}{\pi}\right]$ for one, two or three dimensions. This Kernel shows :

- no interaction for $r > 2h$
- a continous second derivative $\rightarrow$ It is not sensitive to particle disorder

Smoothed Particle Hydrodynamics

The Smoothing Length h

The smoothing length h determines:

- The resolution of particles at a point
- The number of neighbours to contribute to a property at a point

The most common rule for choosing h is:

$$h \propto \frac{1}{\langle\rho\rangle^{\frac{1}{\nu}}}, \qquad \langle\rho\rangle = \frac{1}{n}\sum_b \rho_b$$

With ν: number of dimensions, and n : number of particles. For better accuracy one often chooses: $h = h(t)$. Typical h is calculated from:

$$\frac{dh_a}{dt} = -\left(\frac{h_a}{\nu\rho_a}\right)\frac{d\rho_a}{dt}$$

This increases resolution at points of higher density. But momentum is conserved only for symmetric kernels

Smoothed Particle Hydrodynamics
Error Treatment

- Through tailor expansion we obtain an error of the integral interpolant of $O(h^2)$ for even kernels.
- For the summation interpolant with ordered particles one can estimate the error by the Shoenberg formula. One mostly gets an error of $O(h^2)$.
- If the particles are disordered there is no traditional error order estimate. But the errors are much smaller than Monte Carlo estimate would suggest.

⇒ This is one of the major disadvantages of SPH. There is no easy way to treat errors.

Smoothed Particle Hydrodynamics

Physical Interpretation

To get a 'feeling' for the method we take a look at the vorticity of a particle a :

$$\rho_a \left(\nabla \times \vec{v}\right)_a = \sum_b m_b \vec{v}_{ab} \times \nabla_a W_{ab}$$

For a gaussian kernel as above this gives:

$$\begin{aligned} \rho_a \left(\nabla \times \vec{v}\right)_a &\propto \sum_b m_b \left(\vec{v}_a - \vec{v}_b\right) \times \left(\vec{r} - \vec{r}'\right) W_{ab} \\ &= \sum_b m_b \, L_{ab} \, W_{ab} \end{aligned}$$

where L_{ab} is the angular momentum of the particle a. ⇒ One advantage of SPH is, the physical interpretation often easily follows when using a gaussian kernel

Implementation

In the original SPH the computation time grows with N^2 where N is the number of particles. With the introduction of

- A variable smoothing length
- Individual timesteps for each particle
- The hierarchical tree method for the gravitational potential

modern SPH codes show an $N \log N$ operation count.
Further the SPH code can be nearly completely vectorized for use on modern supercomputer
Code for galaxy formation (GADGET) can be found at:
http://www.mpa-garching.mpg.de/galform/gadget/index.shtml
by V. Springel and N. Yoshida
It contains among others SPH techniques to simulate the baryonic matter.

Equations of Motion
Moving the Particles

One possibility is:

$$\frac{d\vec{r}_a}{dt} = \vec{v}_a$$

But for incompressible fluids the XSPH variant is often used:

$$\frac{d\vec{r}_a}{dt} = \vec{v}_a + \varepsilon \sum_b m_b \left(\frac{\vec{v}_{ba}}{\bar{\rho}_{ab}} \right) W_{ab}$$

with $\bar{\rho}_{ab} = \frac{1}{2} (\rho_a + \rho_b)$ and $0 \leq \varepsilon \leq 1$ as constants
XSPH moves the particles orderly even without viscosity. No dissipation but dispersion is introduced.

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Equations of Motion
The Continuity Equation

In SPH the density of a particle a ρ_a is defined as:

$$\rho_a = \sum_b m_b W_{ab}$$

with this and $\vec{v}_{ab} = \vec{v}_a - \vec{v}_b$ we find the SPH continuity equation:

$$\frac{d\rho_a}{dt} = -\sum_b m_b v_{ab} - \sum_b m_b \frac{f_b}{\rho_b} W_{ab}$$

For a interpretation of this result we find with $m_b = m_b(t)$ and the substantial derivation $\frac{d}{dt}\rho = \frac{\partial}{\partial t}\rho + \rho(\nabla v)$:

$$\frac{\partial}{\partial t} m_b = -m_b \frac{f_b}{\rho_b}$$

This shows: A sink/source is a region where particles loose/gain mass.

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Equations of Motion

The Momentum Equation

We derive the momentum equation in SPH from the Euler equation without external forces; The pressure gradient can be writen as:

$$\frac{d\vec{v}_a}{dt} = -\frac{1}{\rho_a}\sum_b m_b \frac{P_b}{\rho_b}\nabla_a W_{ab}$$

But here linear and angular momentum are not conserved, because the equation is not symmetric in particle exchange. So we symmetrise it This gives us a symmetric momentum equation:

$$\frac{d\vec{v}_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right)\nabla_a W_{ab}$$

For a gaussian kernel we get symmetric central force between particle pairs:

$$m_a \frac{d\vec{v}_a}{dt} = \frac{2 m_a m_b}{h^2}\left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right)\left(\vec{r}_a - \vec{r}_b\right) W_{ab}$$

Equations of Motion
The Thermal Energy Equation

With our interpolation formula we can rewrite the energy equation:

$$\frac{du_a}{dt} = \left(\frac{P_a}{\rho_a^2}\right) \sum_b m_b \vec{v}_{ab} \cdot \nabla_a W_{ab}$$

Or we can out trick again:

$$\frac{du}{dt} = -\nabla \left(\frac{P\vec{v}}{\rho}\right) + \vec{v} \cdot \nabla \left(\frac{P}{\rho}\right) \quad \Rightarrow \quad \frac{du_a}{dt} = \sum_b m_b \left(\frac{P_b}{\rho_b^2}\right) \vec{v}_{ab} \cdot W_{ab}$$

For the symmetrised form we take the average of the two versions:

$$\frac{du_a}{dt} = \frac{1}{2} \sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2}\right) \vec{v}_{ab} \cdot \nabla_a W_{ab}$$

For a gaussian kernel we see $\frac{du_a}{dt} < 0$, this means thermal energy of particle a increases when b aproaches

Viscosity

To model artificial viscosity we introduce the stress tensor Π_{ab}

$$\frac{d\vec{v}_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab} \right) \nabla_a W_{ab}$$

With (for example)

$$\Pi_{ab} = \begin{cases} \frac{-\alpha \bar{c}_{ab} \mu_{ab} + \beta \mu_{ab}^2}{\bar{\rho}_{ab}}, & \vec{v}_{ab} \cdot \vec{r}_{ab} < 0; \\ 0, & \vec{v}_{ab} \cdot \vec{r}_{ab} > 0; \end{cases}$$

$$\mu_{ab} = \frac{h \vec{v}_{ab} \cdot \vec{r}_{ab}}{\vec{r}_{ab}^2 + \eta^2}$$

c_{ab} is speed of sound, $\eta = 0.01\, h^2$ a parameter to prevent singularities, α models shear + bulk viscosity, β models supersonic shocks
This viscosity is:

- Galilean invariant
- vanishes for rigid body rotation
- conserves linear and angular momentum

SPH vs. PPM Method
Adiabatic Collapse of a Isothermal Gas Cloud

We compare SPH with a Particle Particle Mesh algorithm by looking at a simulation of the Collapse of a gas cloud.
It consists of a ideal gas ($\gamma = \frac{5}{3}$) with a density distribution:

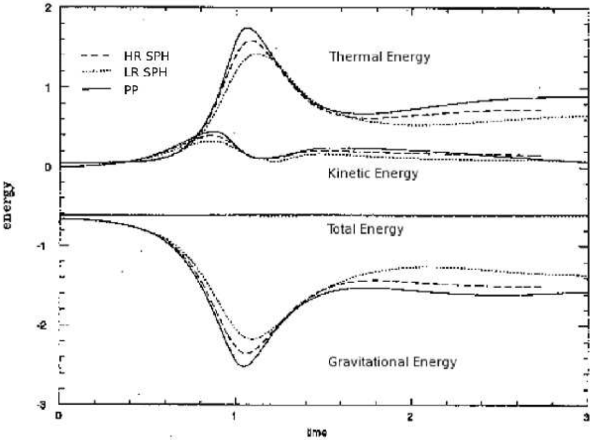
$$\rho(\vec{r}) = \frac{M(R)}{2\pi R^2} \frac{1}{r}$$

where M is the total mass within the cut-off radius R
The model consists of three phases :

- The sphere begins to collapse. Temperature and density rise
- After some time, a central bounce accurs and a shock wave propagates outwards
- For $t \to \infty$ virial equillibrium should be reached. $\Rightarrow \frac{E_{thermal}}{E_{grav}} = -0.5$

We will compare a high resolution SPH (HR SPH: $N = 28768$) and a low resolution SPH (LR SPH: $N = 4224$) with a PPM Method-code using 350 zones.

SPH vs. PPM Method
Energy



For $t < 1$ in the collapse all SPH methods give reasonable results.
At t1 the bounce haPPMens, LR SPH gives high errors, HR SPH is better.
For $t >> 1$ equillibrium is reached, but LR SPH gives a 20% error

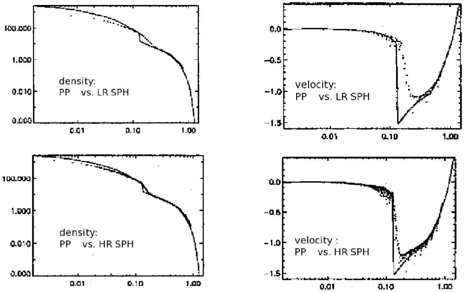
SPH vs. PPM Method

Before Bounce

We now take a look at density ρ and velocity v at $t = 0.77$:

- The density is underestimated by SPH in this stage.
- The discontinuity near $\frac{r}{R*} = 0.1$ ist strongly smoothed out.
- The peak velocity is underestimated by 30%

In smoother regions the results of SPH are significantly better. For all graphs dimensionless units are used.

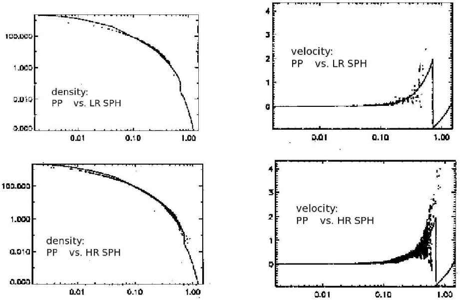


SPH vs. PPM Method

Bounce

At the bounce $t = 1.29$

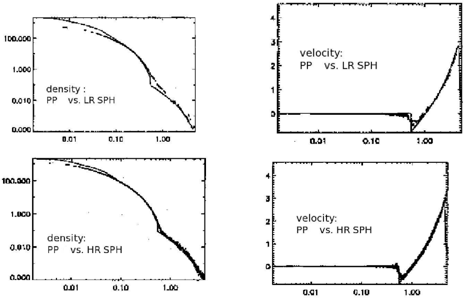
- LR SPH doesn't discribe the discontinuity in outer layers well.
- @ 0.8 the velocity seems to be highly anisotropic
- At outer layers, ρ is small and $\vec{v}$ is high
  $\Rightarrow$ highly supersonic flow



SPH vs. PPM Method
After Bounce

For large times: $t = 1.29$ near thermal equillibrium:

- The system is spherical symmetric
- The HR SPH results agree with the PPM method calculation very well



SPH vs. PPM Method
Results

As a result we can state:

- SPH reproduces the global behaviour of the solution with even a small amount of particles.
- Global properties like thermal, kinetic and gravitational energy were almost quantitatively determined
- For an accurate solution large particle numbers and therefore large computational resources are required
- For accurate local results as much computer power is needed in SPH as in PPM methods.

For more detailed investigation see:
Steinmetz & Müller: On the capabilities and limits of smoothed particle hydrodynamics (1993)

Galaxy Formation
Galaxy Clusters

Galaxies are found distributed in a hierachical way:

- Galaxies form galaxy clusters in scales of up to 10 *Mpc*
- Cluster form superclusters within up to several 100 *Mpc*
- Clusters and superclusters are arranged in the filament structures with large voids

These structures are meant to form from small fluctuations on the homogeneous background.
They are analysed by mostly statistical methods like the Correlation Functions

Galaxy Formation
Numerical Approach

One way to test cosmologies and their behaviour of galaxy formation is to simulate it.
This can be done with pure SPH methods, or with a mixed a PPM roach; PM methods of the dark matter.
But when becomes our simulated gas a galaxy ?
$\Rightarrow$ A distribution of baryonic gas is considered a galaxy if :

- $t_{cooling} < t_{freefall}$ with $t_{cooling} \propto \frac{E}{\dot{E}}$ and $t_{freefall} = \sqrt{\frac{3\pi}{32G\rho}}$
  $\Rightarrow$ Effective cooling
- Its radius is smaller than the jeans length $\lambda_j = c_s \left(\frac{\pi}{G\rho}\right)^{\frac{1}{2}}$
- It shows converging flow: $\nabla \cdot \vec{v} < 0$

A fraction of this mass is then considered 'stellar' mass.

We observe certain statistical properties and compare them with the simulations:

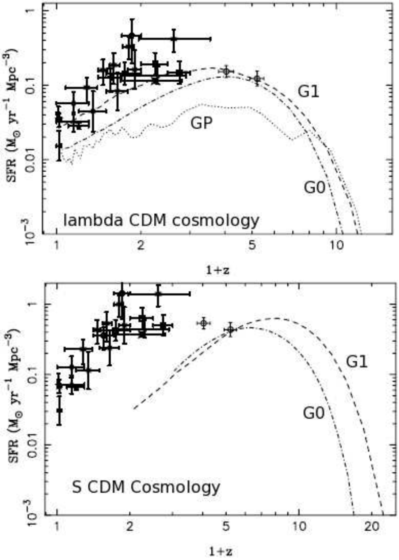
- Correlation Functions
- Star formation rate
- Stellar particle mass function
- Galaxy luminosity functions
- Epoch of formation

As an example we take a look at the Star Formation Rate :

We take a look for 2 different cosmologies: *SCDM* and λ*CDM* and different simulations:

- The λ*CDM* scenario works better than *SCDM*
- For $1 + z < 5$ shock heating takes place: reducing the star formation rate.

G0 and G1 have different resolutions and include: shock heating and radiative cooling Further the GP simulation includes photoionization.

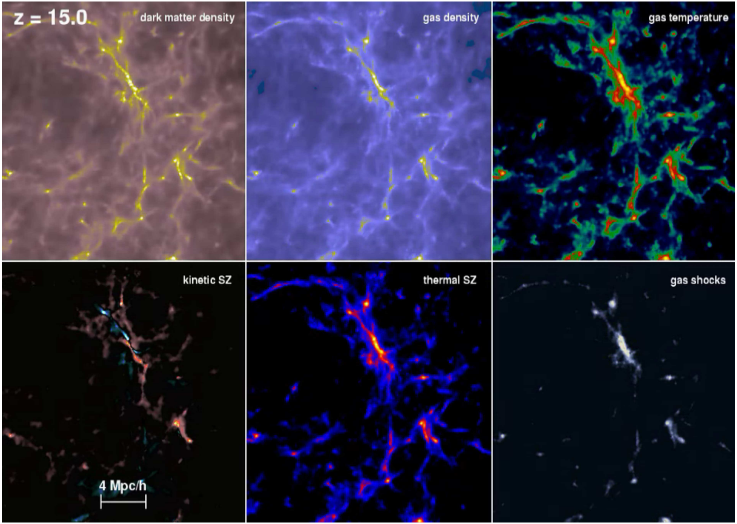


Other Fields of Use

SPH isn't only used in galaxy formation. It is also found in:

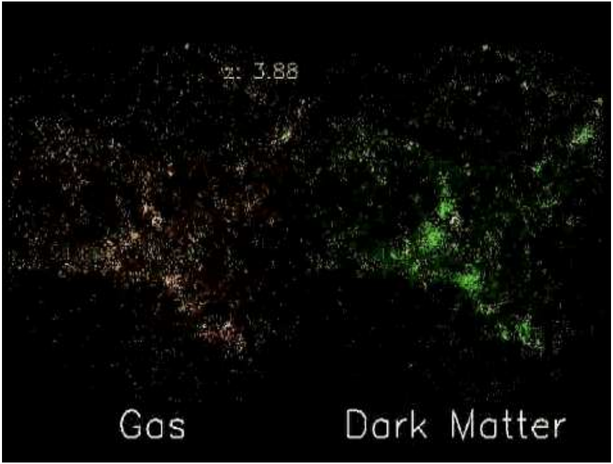
- Gas Dynamics in any way
- Galaxy Mergers
- Binary Stars and Stellar Collisions
- Moon Formation and Impact Problems
- Accretion Disks
- Relativistic Problems; There exists a relativistic SPH version
- Magnetic Phenomena; MSPH code

Movies
GADGET Simulation



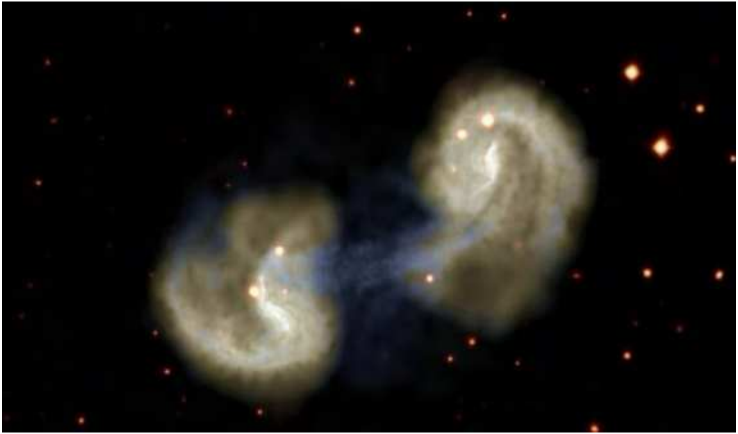
From V. Springel made with the GADGET 2 code:

Movies
Grape SPH Simulation



Gas dynamical simulation of a galaxy with a relatively violent formation history
From M. Steinmetz :

Movies
Galaxy Merger



Two colliding spiral galaxies.
From V. Springel made with the GADGET code:

Advantages and Disadvantages

Advantages:

- Robust code
- Easy to implement
- Easy to interprete (Gaussian kernel)
- Lagrangian
- *N logN* operation count
- gives reasonable global results even with lower resolutions

Disadvantages:

- Sometimes too robust → bugs dont cause an abort of computation
- Bad error handling
- Highly dissipative
- For good local results, much more computation power needed
- Smoothing of discontinuities → Shock handling often problematic

Summary

Smooth Particle Hydrodynamics is

- A nice, easy method to treat hydrodynamics numerically
- Used to simulate galaxy formation with or without PM tree methods treating the gravitational potential

Thanks for your attention!

Literature

- Börner: The Early Universe; Springer 2003
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