
Nuclear Reaction
Networks and
Equilibria
at MPA, Earth, on May 7th
2004 9:32 a.m. +/-5 min

Outline

- What this is all about?
- Coupling of generated energy with hydrodynamic
- Selection of the isotopes
- The CNO-Cycle as an example
- Definition of the network
- Nuclear statistical equilibrium as an approximation

What this is all about?

Why are we doing all this?

- Because we can do it!
- We want the explain the origin of the isotopes
- Nucleossynthesis calculations can give clues to explosion models of supernovae when they are compared with observations
- It gives answers to other questions like, evolution of stars or chemical evolution of galaxies.

Energy generation

- Each reaction generates or consumes energy.
  $\rightarrow$ coupling with hydrodynamic
- Depending on the reaction type, the generated energy may be kinetic or in form of photons or neutrinos. E.g. A(x,y)B, A(x,γ)B, A(x,ν)B reactions.
- The energy in form of photons and neutrinos may or may not escape. This depends on the external conditions (density, temperature) of the astrophysical site.
- If they escape, this leads to an energy loss.

Coupling to hydrodynamic

Due to the energy generation, there exists an additional source term $\rho\dot{\epsilon}$ in the conservation law for the total specific energy:

$$\dot{\epsilon} = h(d(\text{Amount of isotopes}), T)$$

plus network equations, which will be defined later.

$$[\dot{\epsilon}] = \text{erg g}^{-1}\text{s}^{-1}$$

Problems due to the Coupling

- Problems are mainly computational limitations, because solving the network equations ***can*** be very time consu-ming. (This depends on the problem).

- Another problem is due to the fact, that solving the set of hydrodynamical equations ***is*** usually very time consuming.

$\Rightarrow$ you are forced to make some simplifications

Selection of the Isotopes

The first question you have to ask yourself is:
„Do I want to model a hydro static object (e.g. the sun) or do I want to model an explosion (e.g. supernovae)?“

- In the first case the hydrodynamical timescales are long compared to nuclear timescales. $\Rightarrow$ it is a good approximation to neglect some β – unstable isotopes.
- In the second case the hydrodynamical timescales can be compared to nuclear timescales. $\Rightarrow$ you have to consider β – unstable isotopes, because in these cases the cross sections for strong reactions are high enough to be important.

The second Case

- In principle one has to consider more isotopes.
- This is usually not done due to computational limitations

<u>Simplification</u>:
A reduced network is used to get an approximation for the Energy generation.

E.g. in type Ia supernova simulations done at the MPA the "network" consists of the isotopes ^{12}C, ^{16}O, ^{4}He, "^{24}Mg", "^{56}Ni"

How do you get information of the nucleosynthesis in such cases?

- When simulating an explosive event, one puts „tracer particles“ in the simulated fluid.

- These particles record temperature and density

- These information are sufficient to calculate the nucleosynthesis by solving the network equations (will be defined later) in a post process.

The first Case (E.g. CNO-Cycle)

- May start, if sufficient heavy isotopes are available when a star has formed.
- The temperatures are usually of the order of 10^6 K

The basic reactions (CN-Cycle):

$^{12}C(p,\gamma)^{13}N(\beta^+\nu)^{13}C(p,\gamma)^{14}N(p,\gamma)^{15}O(\beta^+\nu)^{15}N(p,\alpha)^{12}C$

weak reactions,
ν escapes

NO-Cycle:

$^{15}N(p,\gamma)^{16}O(p,\gamma)^{17}F(\beta^+\nu)^{17}O(p,\alpha)^{14}N$

Isotope	Lifetime
^{13}N	870 sec
^{15}O	178 sec
^{17}F	95 sec

Lifetime of Isotope X

Definition: Lifetime of Isotope X $\tau(X) := \frac{X}{dX/dt} = \frac{X}{\dot{X}}$

Now consider a reaction: X(p,γ)Y

You can calculate the lifetime of X against proton capture

$$\tau(X) = \frac{1}{p \times \lambda_{pX}}$$

Where p stands for the number of protons per unit volume

and λ_{pX} is the temperature dependent (!) cross section for the reaction.

$\Rightarrow$ if τ is small, then a reaction is very likely

if τ is large, then a reaction may be neglected

How to model the CNO-Cycle?

$$
\begin{aligned}
d^{12}\mathrm{C}/dt &= \lambda_{p15}\mathrm{H}^{15}\mathrm{N} - \lambda_{p12}\mathrm{H}^{12}\mathrm{C} \\
&= {}^{15}\mathrm{N}/\tau_{15} - {}^{12}\mathrm{C}/\tau_{12} + \\
d^{13}\mathrm{N}/dt &= {}^{12}\mathrm{C}/\tau_{12} - {}^{13}\mathrm{N}/\tau_{\beta}(13) - {}^{13}\mathrm{N}/\tau_{p}({}^{13}\mathrm{N}) \\
d^{13}\mathrm{C}/dt &= {}^{13}\mathrm{N}/\tau_{\beta}(13) - {}^{13}\mathrm{C}/\tau_{13} \\
d^{14}\mathrm{N}/dt &= {}^{13}\mathrm{C}/\tau_{13} - {}^{14}\mathrm{N}/\tau_{14} + {}^{17}\mathrm{O}/\tau_{17} \\
d^{15}\mathrm{O}/dt &= {}^{14}\mathrm{N}/\tau_{14} - {}^{15}\mathrm{O}/\tau_{\beta}(15) - {}^{15}\mathrm{O}/\tau_{p}({}^{15}\mathrm{O}) \\
d^{15}\mathrm{N}/dt &= {}^{15}\mathrm{O}/\tau_{\beta}(15) - {}^{15}\mathrm{N}/\tau_{15} \\
d^{16}\mathrm{O}/dt &= {}^{15}\mathrm{N}/\tau_{15} - {}^{16}\mathrm{O}/\tau_{16} \\
d^{17}\mathrm{F}/dt &= {}^{16}\mathrm{O}/\tau_{16} - {}^{17}\mathrm{F}/\tau_{\beta}(17) - {}^{17}\mathrm{F}/\tau_{p}({}^{17}\mathrm{F}) \\
d^{17}\mathrm{O}/dt &= {}^{17}\mathrm{F}/\tau_{\beta}(17) - {}^{17}\mathrm{O}/\tau_{17}
\end{aligned}
$$

Due to the fact that

$\tau_p >> \tau_\beta$

Only for temperatures of the order of 10^6 K !

Further simplifications

Consider the equation $$\frac{d^{13}\mathrm{N}}{dt} = \frac{^{12}C}{\tau_{12}} - \frac{^{13}N}{\tau_\beta(13)}$$

Suppose ^{12}C and the τ's are constant over sufficiently long period of time.

Then this equation is self regulating in the sense that the left side is >0, if there is not enough ^{13}N and vice versa.

⇒ Is there a state where the left side is =0?

YES! At least as a very good approximation.

Further Simplifications

A solution of $\frac{d^{13}\mathrm{N}}{dt} = \frac{^{12}C}{\tau_{12}} - \frac{^{13}N}{\tau_\beta(13)}$ with the mentioned assumptions is given by

$$^{13}N(t) = \frac{\tau_\beta(13)}{\tau_{12}} {}^{12}C(1 - e^{-t/\tau_\beta(13)})$$

⇒ This equation reaches an equilibrium value (dN/dt=0) of (^{13}N/^{12}C)=τ_β(13)/τ_{12} in times of the order of τ_β

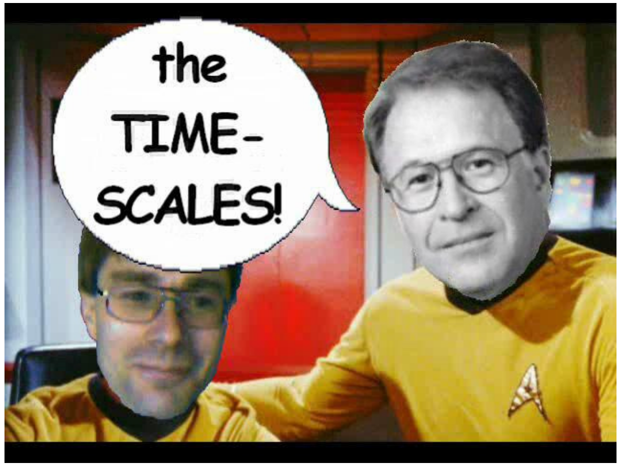
⇒ In times of the order of minutes it is possible to eliminate the Isotopes ^{13}N, ^{15}O and ^{17}F from the system that describes the CNO-cycle

Result of the Simplification

$$
\begin{aligned}
d^{12}\mathrm{C}/dt &= {}^{15}\mathrm{N}/\tau_{15} - {}^{12}\mathrm{C}/\tau_{12} \\
d^{13}\mathrm{C}/dt &= {}^{12}\mathrm{C}/\tau_{12} - {}^{13}\mathrm{C}/\tau_{13} \\
d^{14}\mathrm{N}/dt &= {}^{13}\mathrm{C}/\tau_{13} - {}^{14}\mathrm{N}/\tau_{14} + {}^{17}\mathrm{O}/\tau_{17} \\
d^{15}\mathrm{N}/dt &= {}^{14}\mathrm{N}/\tau_{14} - {}^{15}\mathrm{N}/\tau_{15} \\
d^{16}\mathrm{O}/dt &= {}^{15}\mathrm{N}/\tau_{15} - {}^{16}\mathrm{O}/\tau_{16} \\
d^{17}\mathrm{O}/dt &= {}^{16}\mathrm{O}/\tau_{16} - {}^{17}\mathrm{O}/\tau_{17}
\end{aligned}
$$

When you model non explosive astrophysical sites you can usually make some simplifications for your network

BUT



Definition of a Network

- The CNO-Cycle is just an example of a network.
- The isotopes in here are p and some C,N,O – isotopes; a relatively small number

- The relevant reactions are proton capture reactions and weak decays.

In general one has one deal with a lot more reactions like α captures, photo disintegrations, heavy ion reactions, “weak capture reactions”, decays and three body reactions ….

Definition of a Network

- If you want to model all this reactions, you have to write down a more general set of differential equations than it was done for the CNO-Cycle.

Suppose you want to consider *m* different isotopes. And let n_i be the number of isotope i per unit volume, then we Obtain the following equation:

Definition of a Network

$$\dot{n}_i = \sum_j c_i^j \lambda_i^j n_j + \sum_{j,k} c_i^{j,k} < j,k > n_j n_k + \sum_{j,k,l} c_i^{j,k,l} < j,k,l > n_j n_k n_l$$

Statistical factors to prevent double counting

$$c_i^j := \pm N_i$$

$$c_i^{j,k} := \pm \frac{N_i}{N_j! N_k!}$$

$$c_i^{j,k,l} := \pm \frac{N_i}{N_j! N_k! N_l!}$$

$$\begin{aligned}
d^{12}\mathrm{C}/dt &= {}^{15}\mathrm{N}/\tau_{15} - {}^{12}\mathrm{C}/\tau_{12} \\
d^{13}\mathrm{C}/dt &= {}^{12}\mathrm{C}/\tau_{12} - {}^{13}\mathrm{C}/\tau_{13} \\
d^{14}\mathrm{N}/dt &= {}^{13}\mathrm{C}/\tau_{13} - {}^{14}\mathrm{N}/\tau_{14} + {}^{17}\mathrm{O}/\tau_{17} \\
d^{15}\mathrm{N}/dt &= {}^{14}\mathrm{N}/\tau_{14} - {}^{15}\mathrm{N}/\tau_{15} \\
d^{16}\mathrm{O}/dt &= {}^{15}\mathrm{N}/\tau_{15} - {}^{16}\mathrm{O}/\tau_{16} \\
d^{17}\mathrm{O}/dt &= {}^{16}\mathrm{O}/\tau_{16} - {}^{17}\mathrm{O}/\tau_{17}
\end{aligned}$$

Difficulties with this Form

$$\dot{Y}_i = \sum_j c_i^j \lambda_i^j Y_j + \sum_{j,k} c_i^{j,k} \rho N_A < j,k > Y_j Y_k + \sum_{j,k,l} c_i^{j,k,l} (\rho N_A)^2 < j,k,l > Y_j Y_k Y_l$$

- When you want to calculate a solution for an explosion, you encounter the problem, that total number of isotopes is changing. $\rightarrow$ you would run into unnecessary bookkeeping

- you have to introduce some new quantities

a) $\rho := (\sum_i n_i A_i)/N_A$

b) Mass fraction: $X_i := (n_i A_i)/(\rho N_A)$ with $\sum_i X_i = 1$

c) Specific abundances: $Y_i := X_i/A_i = n_i/(\rho N_A)$

- non linear & stiff $\Rightarrow$ numerical solution is not trivial

- these equations define the nuclear network!

Summary 1

What do we know?

- We know that the last equation defines a nuclear network
- Depending on the situation (hydrostatic situation or explosion) the size of the network can vary.
- When you model the CNO-Cycle, 10 isotopes are sufficient
- But in case of (supernova) explosions you need about a factor of 30 to 50 more isotopes! (only in a post process)
- The energy generation of the network is coupled to hydrodynamic. $\rightarrow$ A reduced network is often used

The NSE

- The NSE is characterized by the fact that all strong reactions are in an equilibrium

$$A+x \rightleftharpoons B+y$$

  where A,B,x,y can be every kind of particle!

- This can occur if the temperatures are so high that basically all Coulomb barriers can be overcome. (densities above 10^7 g/cm^{-3} assumed).
- This means corresponding reaction rates are identical (up to $\pm$ 1). $\rightarrow$ The network equations are replaced.

The NSE

- If you assume that ALL strong reactions are in an equilibrium, you may write for the chem. potential μ of a nucleus (A,Z) with Z protons and A-Z neutrons

$$\mu(A,Z)=Z\mu_p + (A-Z)\mu_n + B(A,Z)$$

where $B(A,Z)=c^2(ZM_h+(A-Z)M_n-M(A,Z))$ is the binding energy of the nucleus (A,Z)

- Assuming a non relativistic Boltzmanngas you obtain for the number of nucleus (A,Z) per unit volume

$$n(A,Z)=G(A,Z)A^{3/2}h^{-3}(2\pi\, M_u\, kT)^{3/2}\exp(\beta\mu(A,Z))$$

The NSE

Using the last equation for the nucleus, protons and neutrons and the relation for the chemical you will get

$$n(A,Z) = G(A,Z)A^{3/2}\frac{n_p^Z n_n^{A-Z}}{2^A}\left(\frac{(2\pi M_u kT)^{\frac{3}{2}}}{h^3}\right)^{1-A}\exp\left(\frac{B(A,Z)}{kT}\right)$$

- The number densities are fixed by the neutron to proton ratio, the binding energy and the temperature.
- The neutron to proton ratio can only be changed by weak interactions.
- A “weak network” is solved first, then together with mass and charge conservation the new number densities are calculated.

Literature

- „Principles of Stellar Evolution and Nucleosynthesis" – D.D. Clayton 1983 Univ. Chicago Press.
- „Nucleosynthesis and Supernovae" – W.D. Arnett 1996 Princeton Univ. Press
- Thielemann et. al. Astro-ph/9802077 „Nucleosynthesis Basics and Applications to Supernovae
- Hillebrandt & Niemeyer „Type Ia Supernova Explosion Models" Annu. Rev. As. Aphy. 2000. 38:191-230
- References in literature mentioned above