Dark Matter with Phase Space Elements

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Abel, Hahn, Kaehler (2012), MNRAS
Kaehler, Hahn, Abel (2012), IEEE TVCG
Hahn, Abel, Kaehler (2013), MNRAS
Angulo, Hahn, Abel (2013), MNRAS
Hahn, Angulo, Abel (2014), MNRAS subm.
Hahn & Angulo (2015), MNRAS subm.

What is Dark Matter?

microscopic ⟶ continuum limit

proton = 1GeV, WIMP 100GeV? -> 10^{21}/g

cold (or at most lukewarm) ⟶ $v_{thermal} \ll v_{bulk}$

e.g. thermally produced at very early times, cooled since then

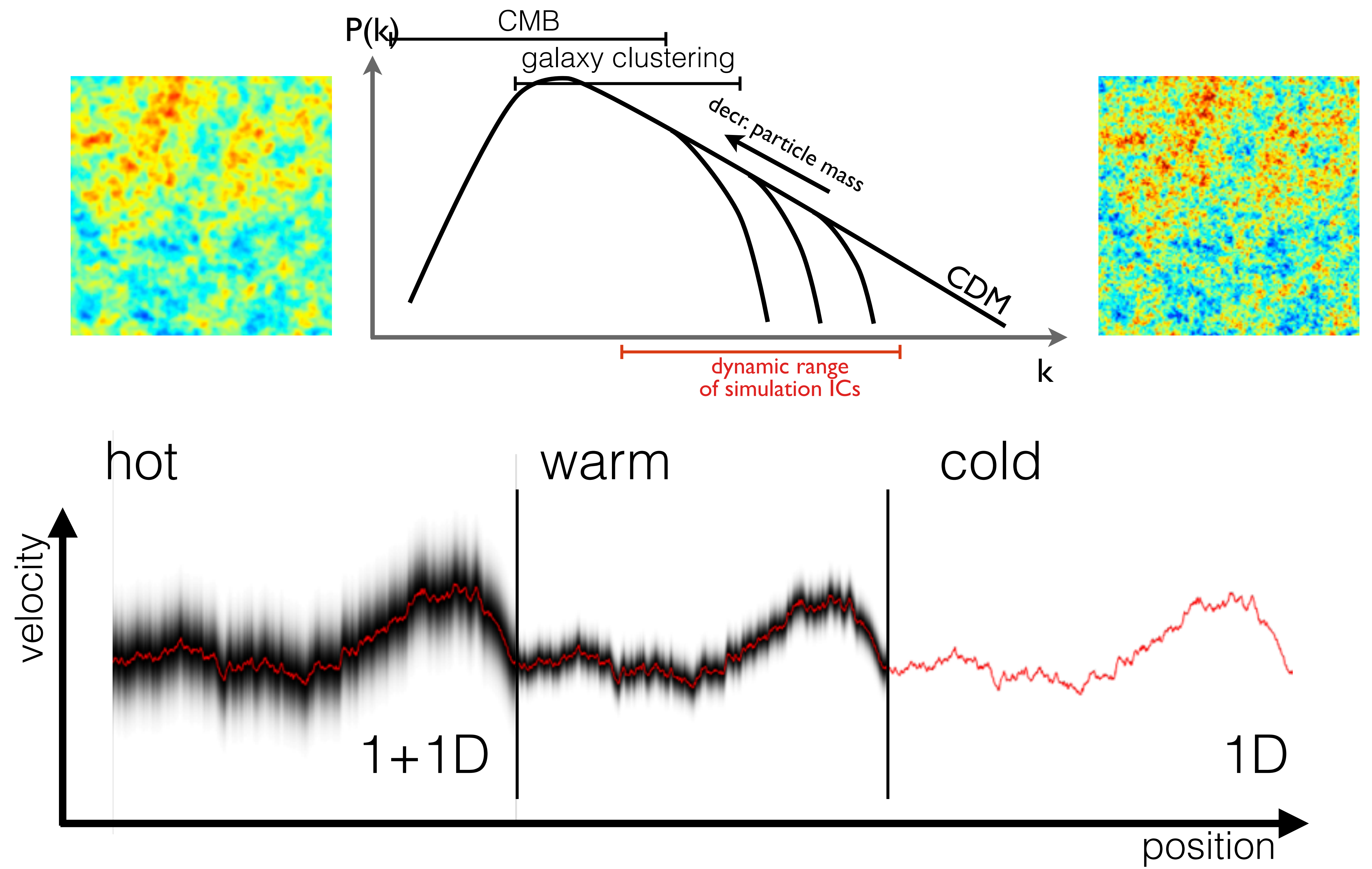
negligible cross-section ⟶ $\sigma_{DM} \ll \sigma_{em}$ collisionless

weak-scale or even weaker

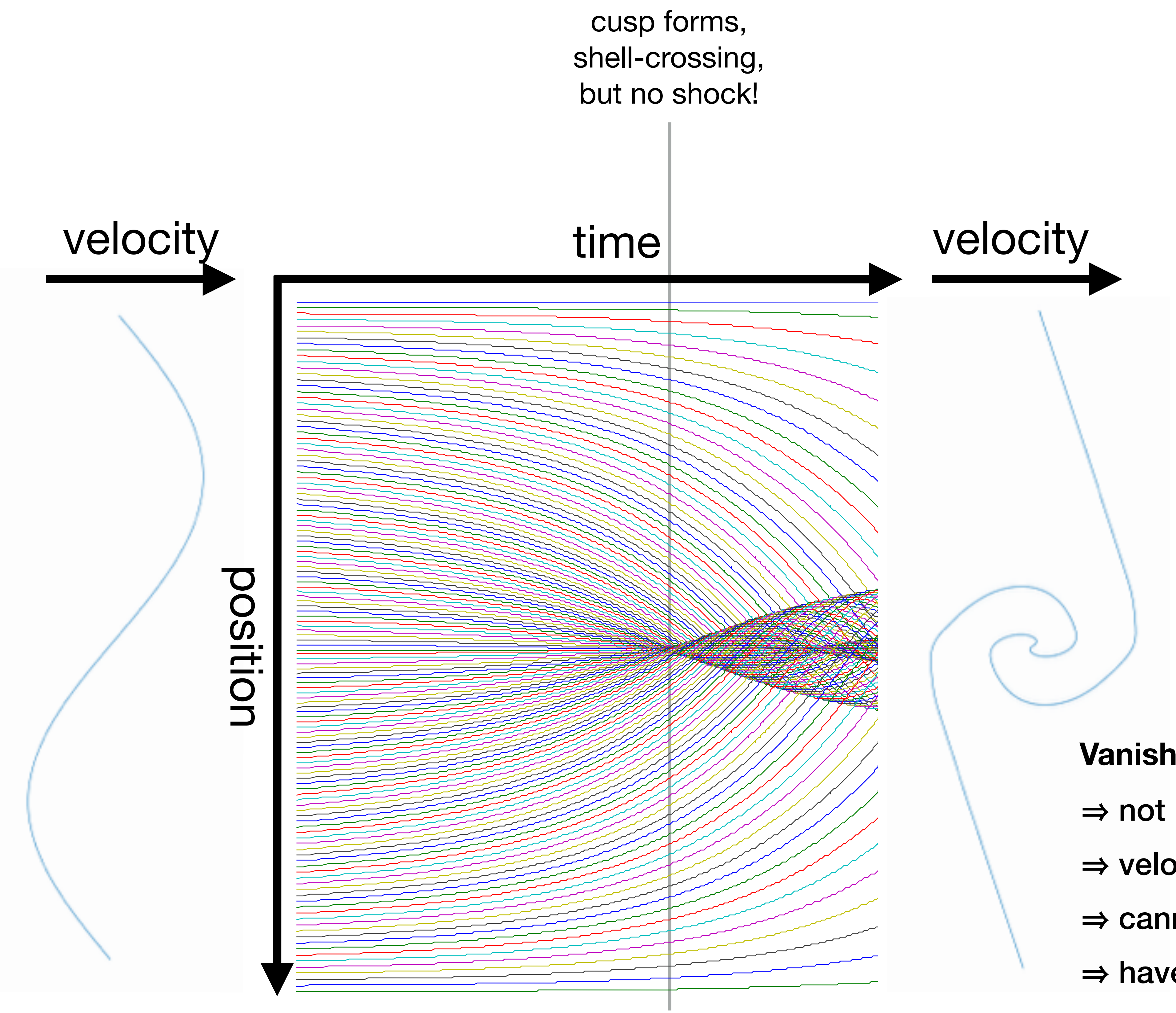
…and also the dominant gravitating component (~80%)

at first order, structure formation is well described by assuming all matter is dark matter

Dark Matter - properties on small scales



1D behaviour under self-gravity



Vanishing collision-term

⇒ not in hydro limit

⇒ velocity can be multi-valued

⇒ cannot stop at low order moments

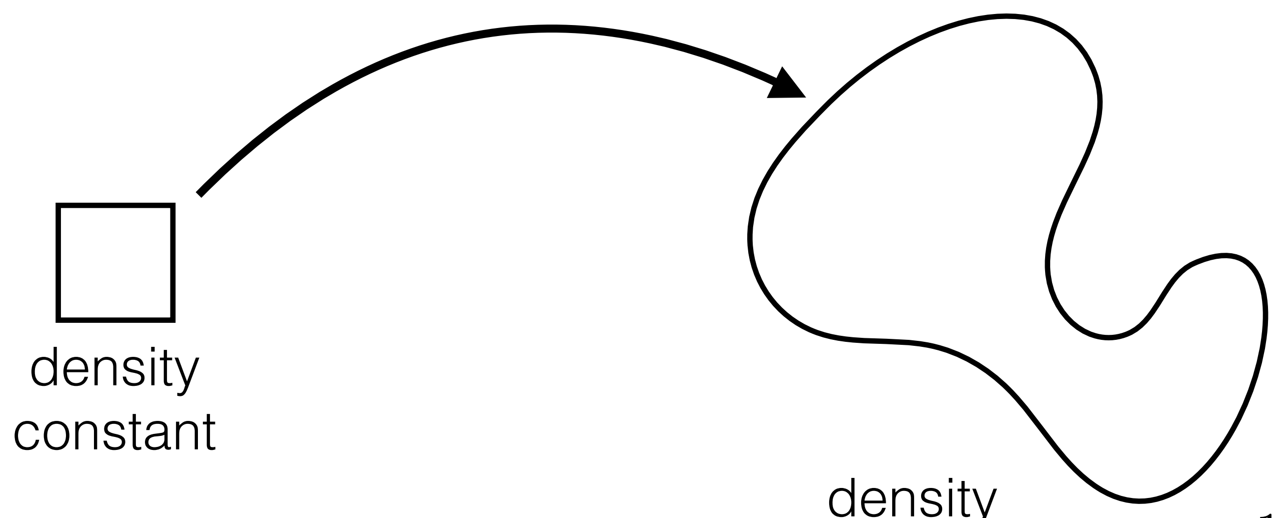
⇒ have to discretize distribution function

⇒ singular caustics emerge

Dark Matter - fluid flow

Lagrangian description, evolution of fluid element

$$\mathbb{Q} \subset \mathbb{R}^3 \to \mathbb{R}^6 \, : \, \mathbf{q} \mapsto (\mathbf{x_q}(t), \mathbf{v_q}(t))$$



$$\rho = m_{\mathrm{DM}} \left| \frac{\partial x_i}{\partial q_j} \right|^{-1}$$

For DM, motion of any point **q** depends only on gravity

$$(\dot{\mathbf{x}}_\mathbf{q}, \dot{\mathbf{v}}_\mathbf{q}) = (\mathbf{v_q}, -\boldsymbol{\nabla}\phi)$$

unlike hydro, no internal temperature, entropy, pressure

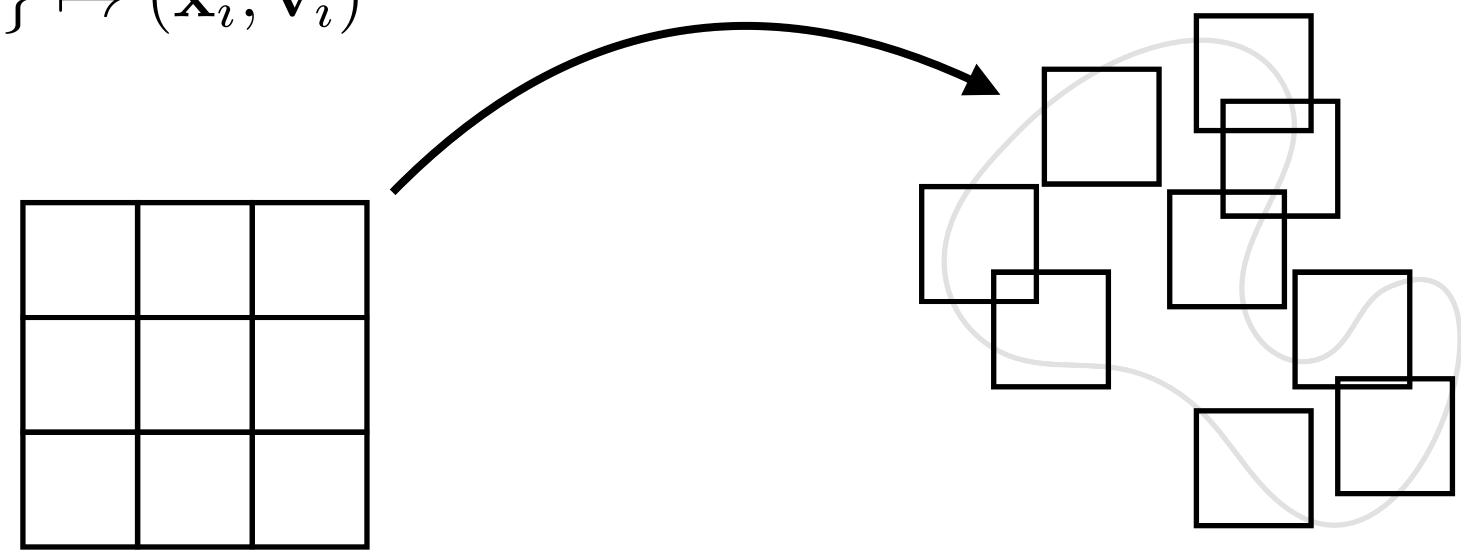
So the quest is to solve Poisson's equation

$$\Delta\phi = 4\pi G\rho$$

N-body vs. continuum approximation

The N-body approximation:

$$i \in \{1 \ldots N\} \mapsto (\mathbf{x}_i, \mathbf{v}_i)$$



⇒ EoM are just Hamiltonian N-body eq. (method of characteristics)

for small N, density field is poorly estimated,

$$\rho = m_p \sum \delta_D(x - x_i) \otimes W$$

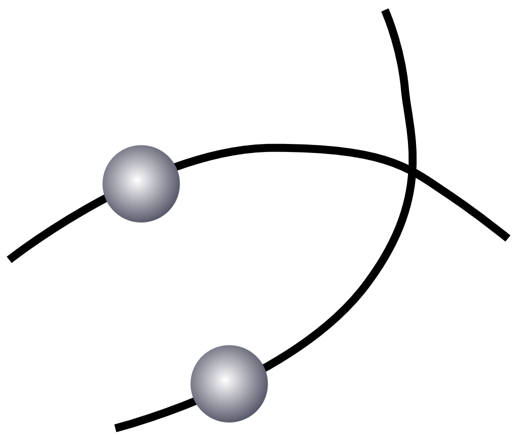
continuum structure is given up, but 'easy' to solve for forces

hope that as N->very large numbers, approach collisionless continuum

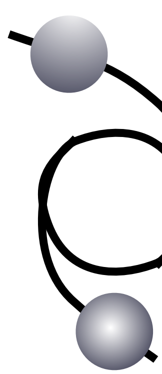
Problems of the N-body method

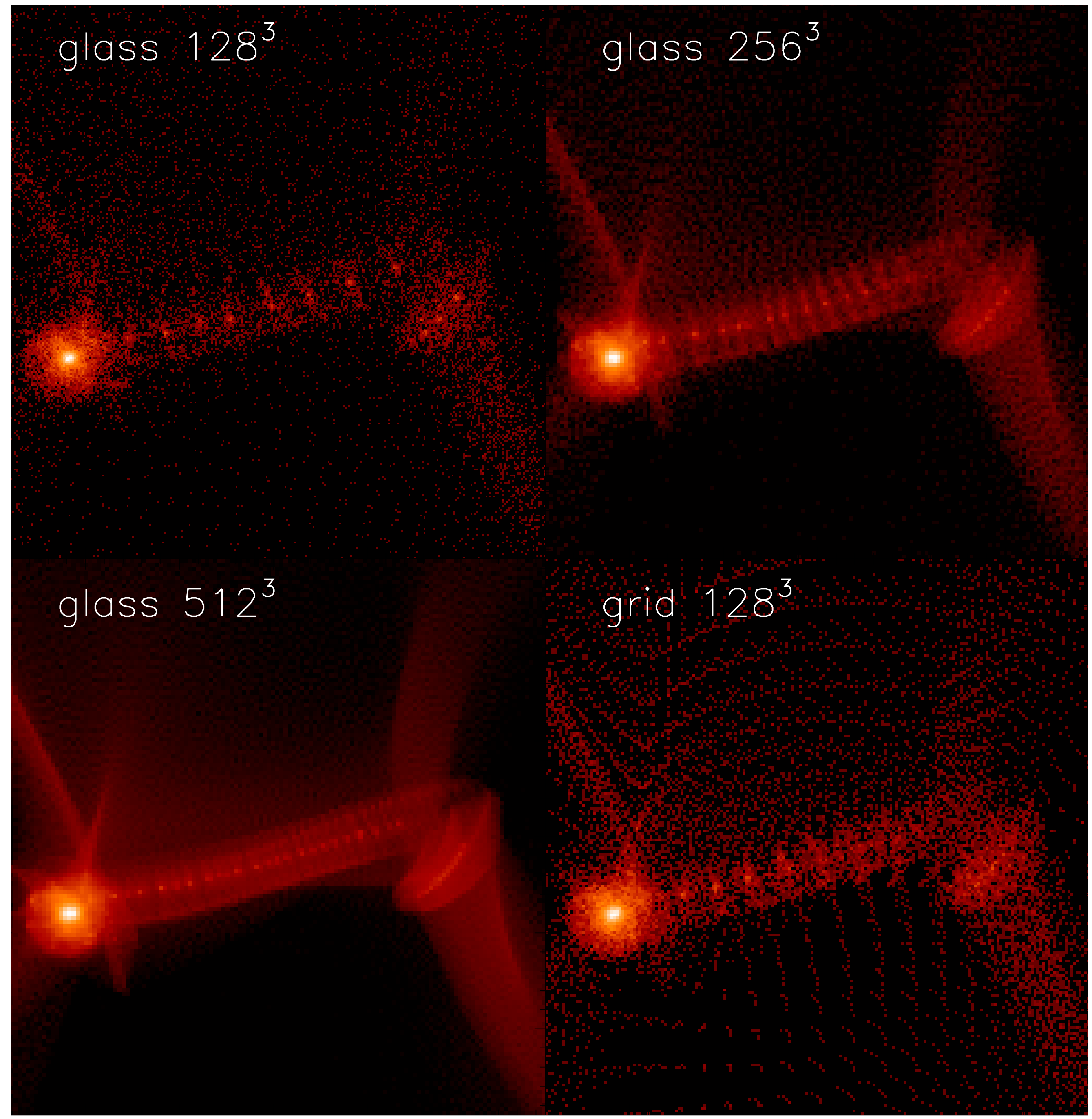
discreteness effects with some influence of softening

Scattering



Clumping/
Fragmentation





Wang&White 2007

Most obvious for non-CDM simulations!

(e.g. Centrella&Melott 1983, Melott&Shandarin 1989, Wang&White 2007)

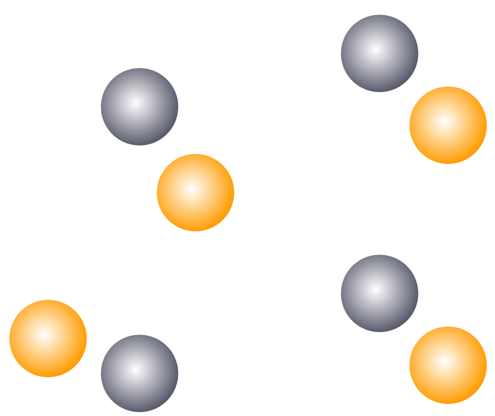
Problems of the N-body method: multi-fluid

Main Problem: two-body effects couple particles!

two fluids, coupled only through gravity:

$$\frac{\partial f_{1,2}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}} f_{1,2} - \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_{1,2} = 0$$

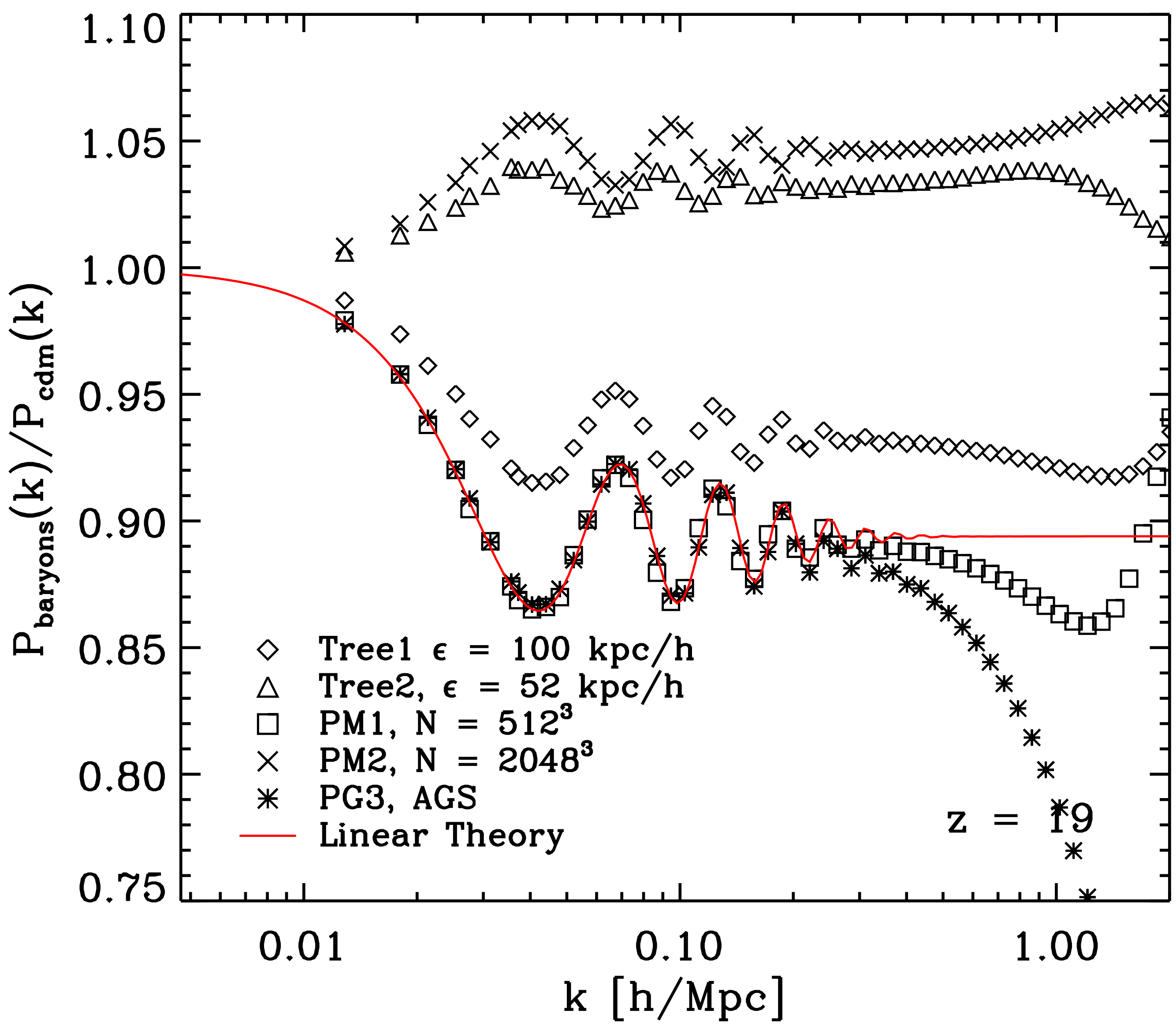
$$\Delta\phi = 4\pi G\,(\rho_1 + \rho_2)$$



very sensitive to spurious coupling!

Problem for precision predictions of high-z baryon distribution

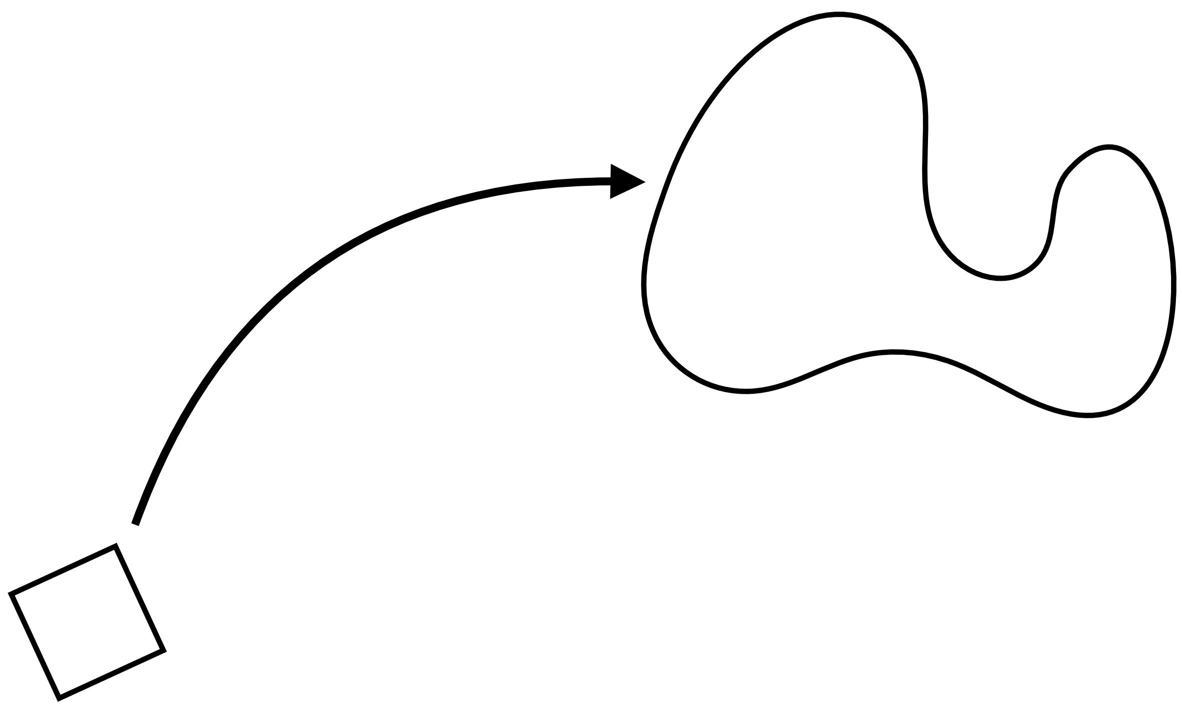
Angulo, Hahn & Abel 2013

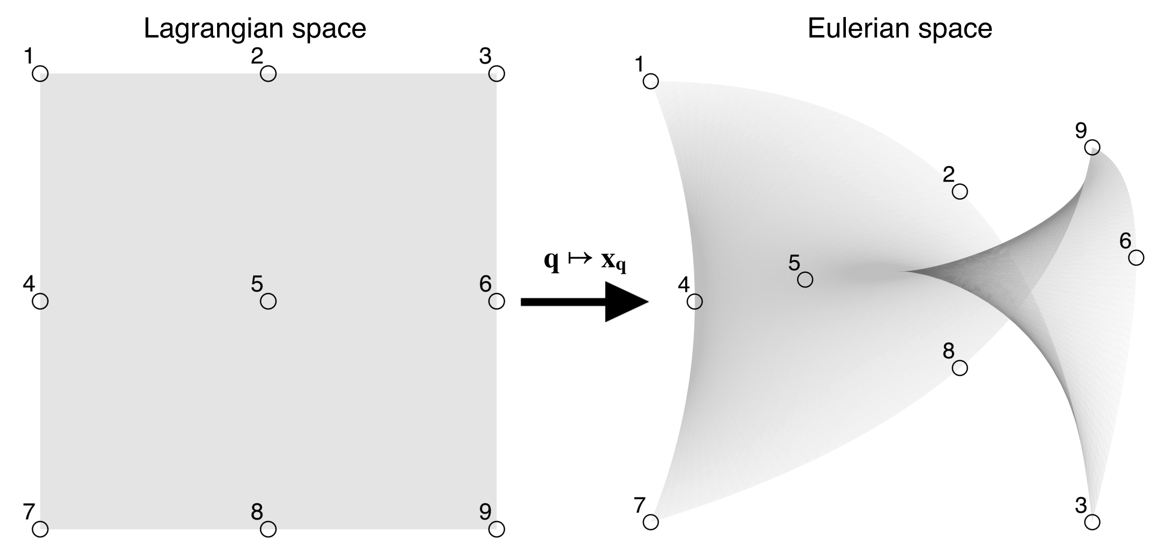


Dark Matter - fluid flow, full manifold description

Lagrangian description, evolution of fluid element

$$\mathbb{Q} \subset \mathbb{R}^3 \to \mathbb{R}^6 \;:\; \mathbf{q} \mapsto (\mathbf{x_q}(t), \mathbf{v_q}(t))$$





Describe map between Lagrangian and Eulerian space by (infinite dimensional) space of tri-polynomials

$$Q \in P_k = \{\pi(\mathbf{q}) \mid \pi(\mathbf{q}) = \sum_{\alpha,\beta,\gamma=0}^{k} a_{\alpha\beta\gamma}\, q_0^\alpha q_1^\beta q_2^\gamma\}$$

Exact for $k \to \infty$, manifold tracking instead of particles

Equations of motion:

N-body characteristics

$$\dot{\mathbf{x}}_i = \mathbf{v}_i, \quad \text{and} \quad \dot{\mathbf{v}}_i = -\left.\boldsymbol{\nabla}_x \phi\right|_{\mathbf{x}_i}, \quad \text{with } \; i \in \mathbb{N}$$

Characteristics on Lagrangian manifold

$$\dot{\mathbf{x}}_\mathbf{q} = \mathbf{v}_\mathbf{q}, \quad \text{and} \quad \dot{\mathbf{v}}_\mathbf{q} = -\left.\boldsymbol{\nabla}_x \phi\right|_{\mathbf{x}_\mathbf{q}}, \quad \text{with } \mathbf{q} \in \mathcal{Q}$$

Polynomial expansion of EoM leads to EoM for coefficients

$$\dot{\mathbf{x}}_{\alpha\beta\gamma} = \mathbf{v}_{\alpha\beta\gamma}, \quad \dot{\mathbf{v}}_{\alpha\beta\gamma} = -\rho^{-1}\mathbf{f}_{\alpha\beta\gamma}, \quad \alpha, \beta, \gamma \in \mathbb{N}$$

finite expansion at order *k* leads to the following truncation error:

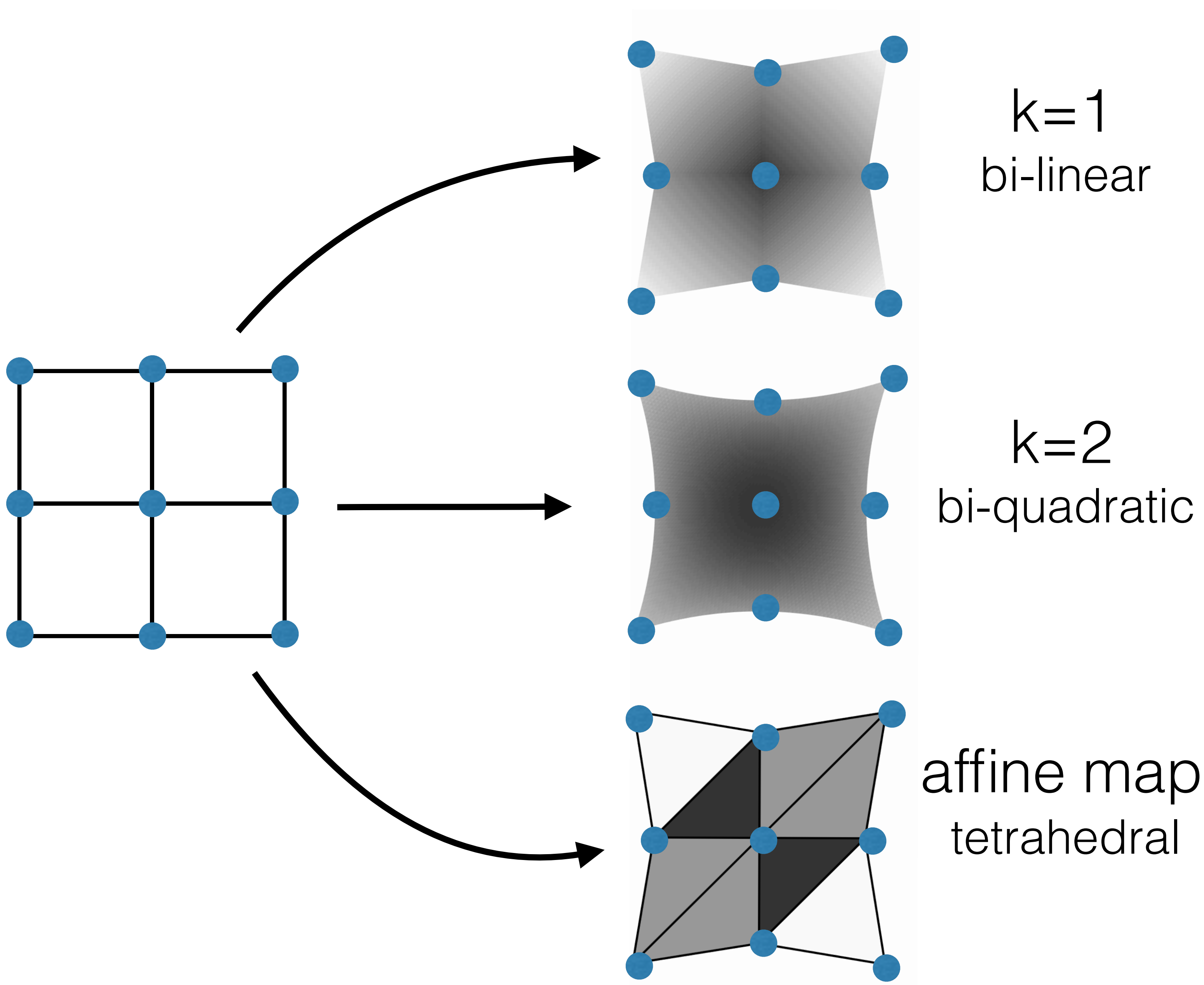
$$\Delta\dot{\mathbf{v}} = -\rho^{-1} \sum_{\alpha,\beta,\gamma=k+1}^{\infty} \mathbf{f}_{\alpha\beta\gamma}\, q_0^\alpha q_1^\beta q_2^\gamma$$

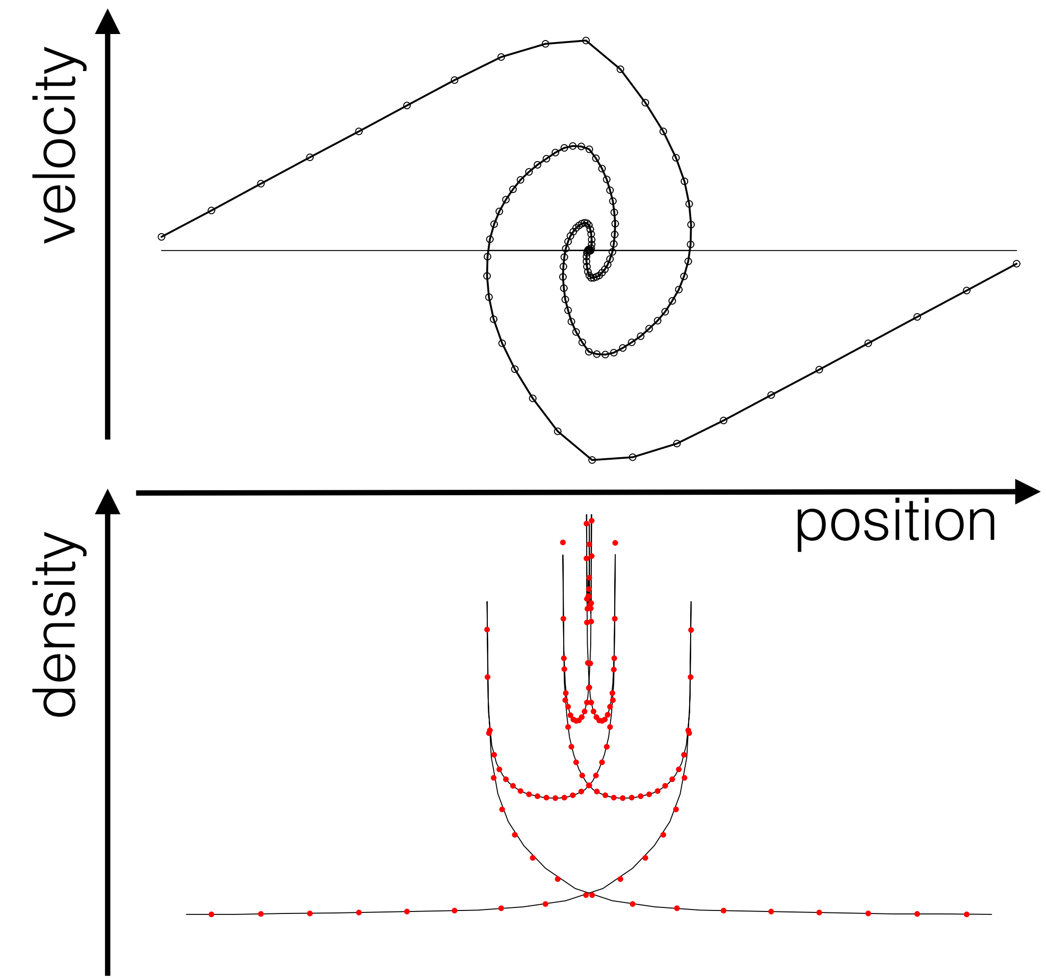
sourced by high order derivatives of the force field across the element

-> need to keep bounded to keep energy conservation bounded
-> refinement essential!

Lagrangian elements of order k

Finite order maps:

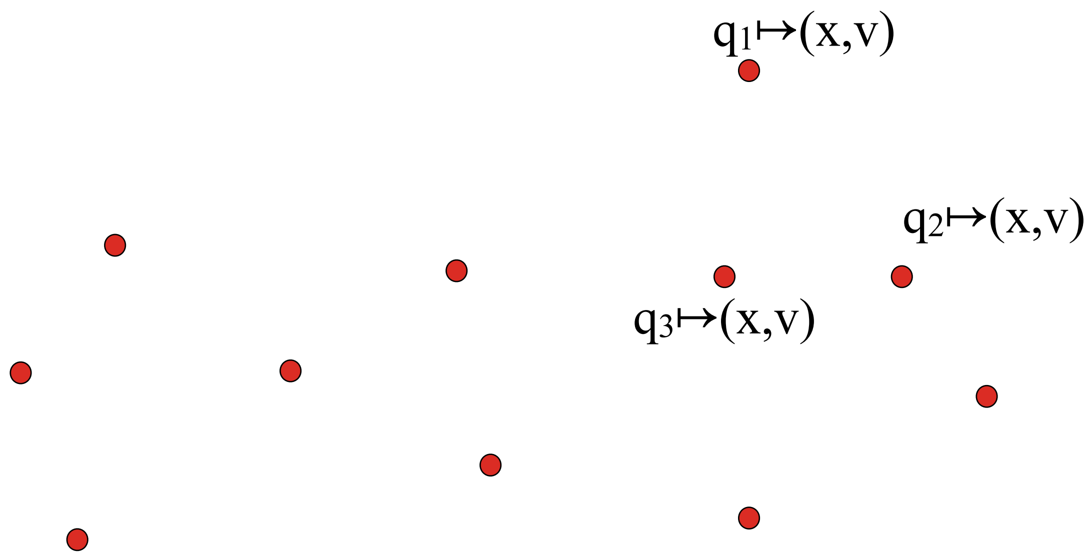


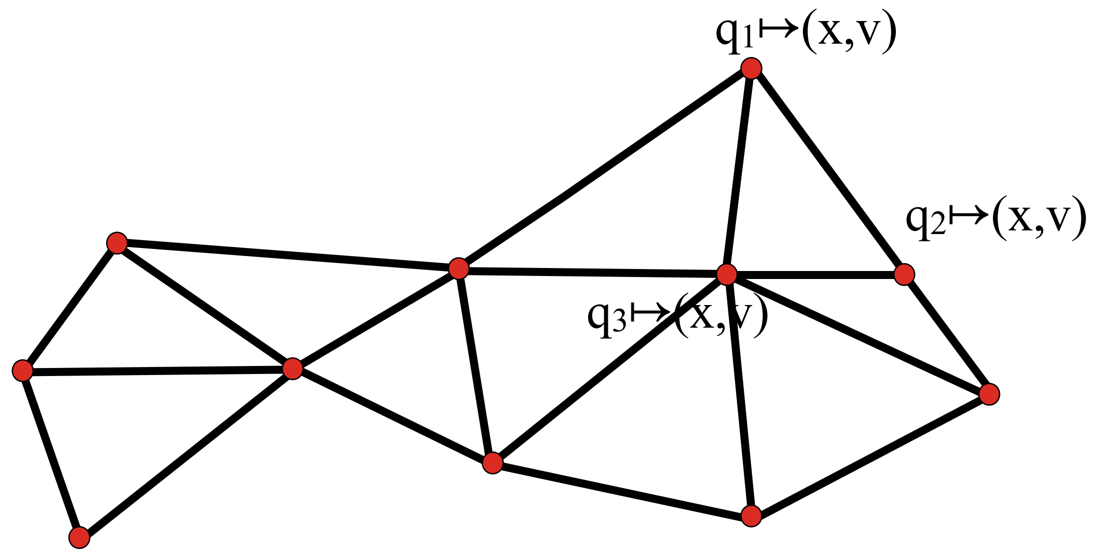


$$\rho = m_{\mathrm{DM}} \left| \frac{\partial x_i}{\partial q_j} \right|^{-1}$$

cost:
truncation error
in EoM!

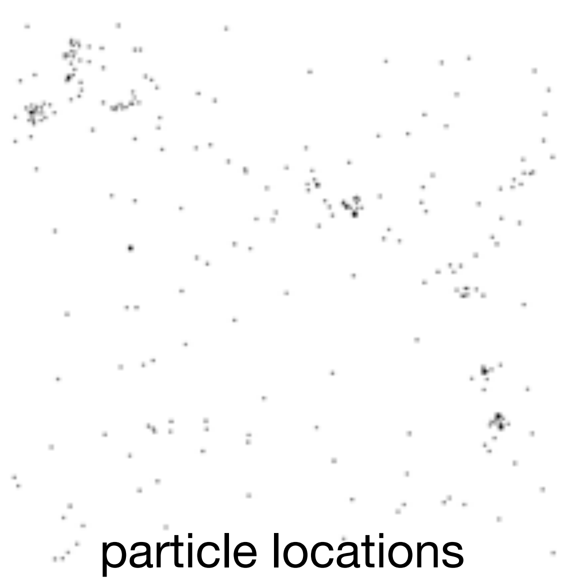
Describing the density field & softening I

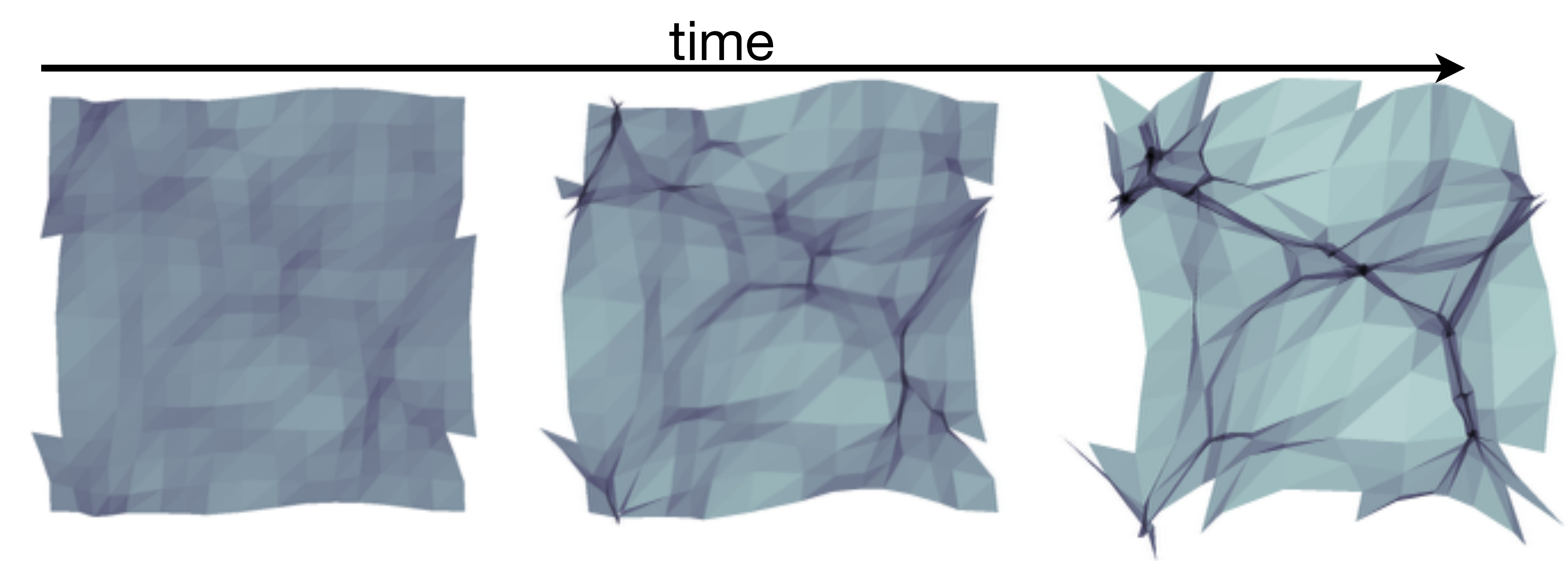




$$\rho = m_p \sum \delta_D(x - x_i) \otimes W$$

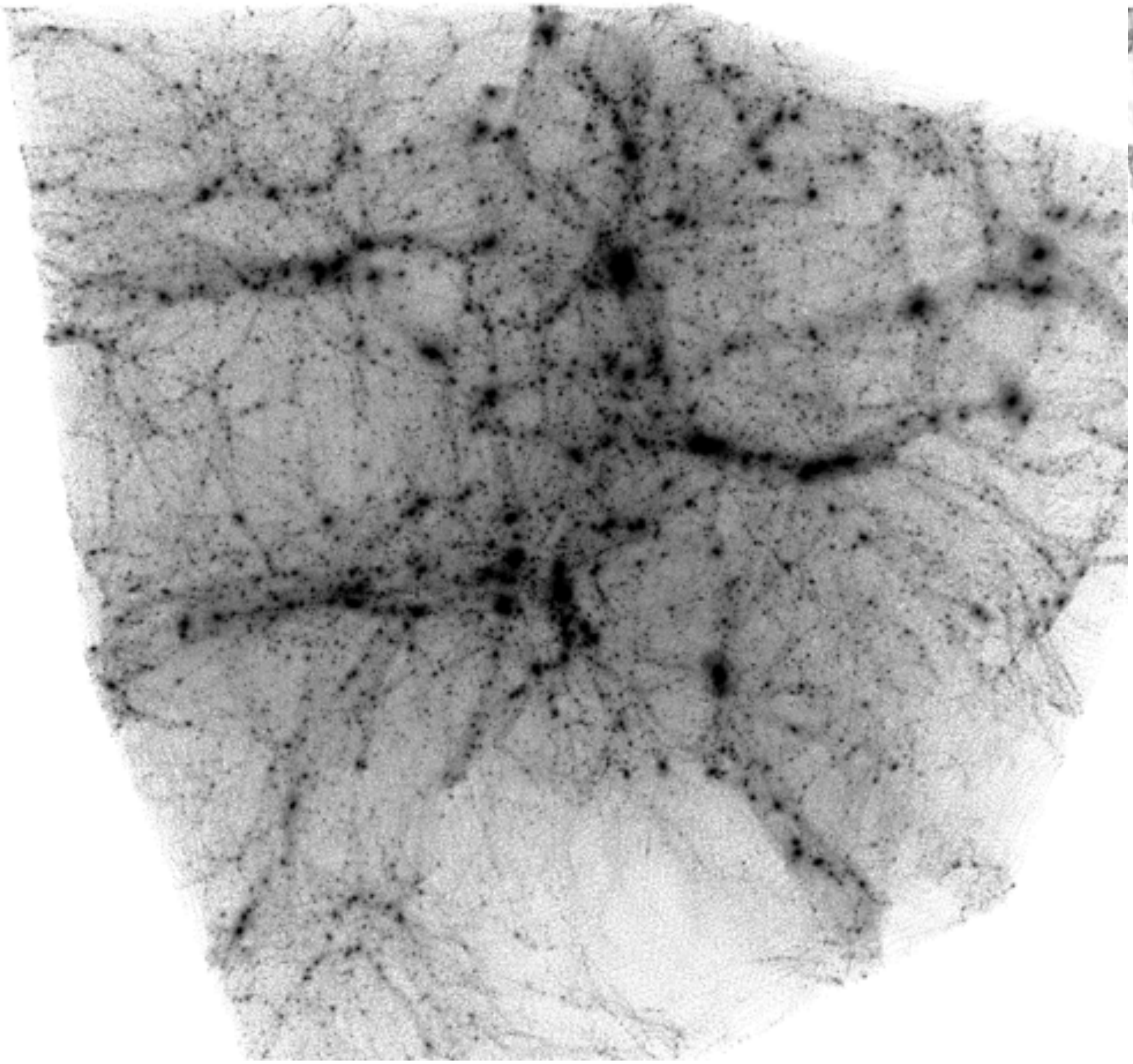
$$\rho = m_p \sum_{\text{streams}} \left| \det \frac{\partial x_i}{\partial q_j} \right|^{-1}$$



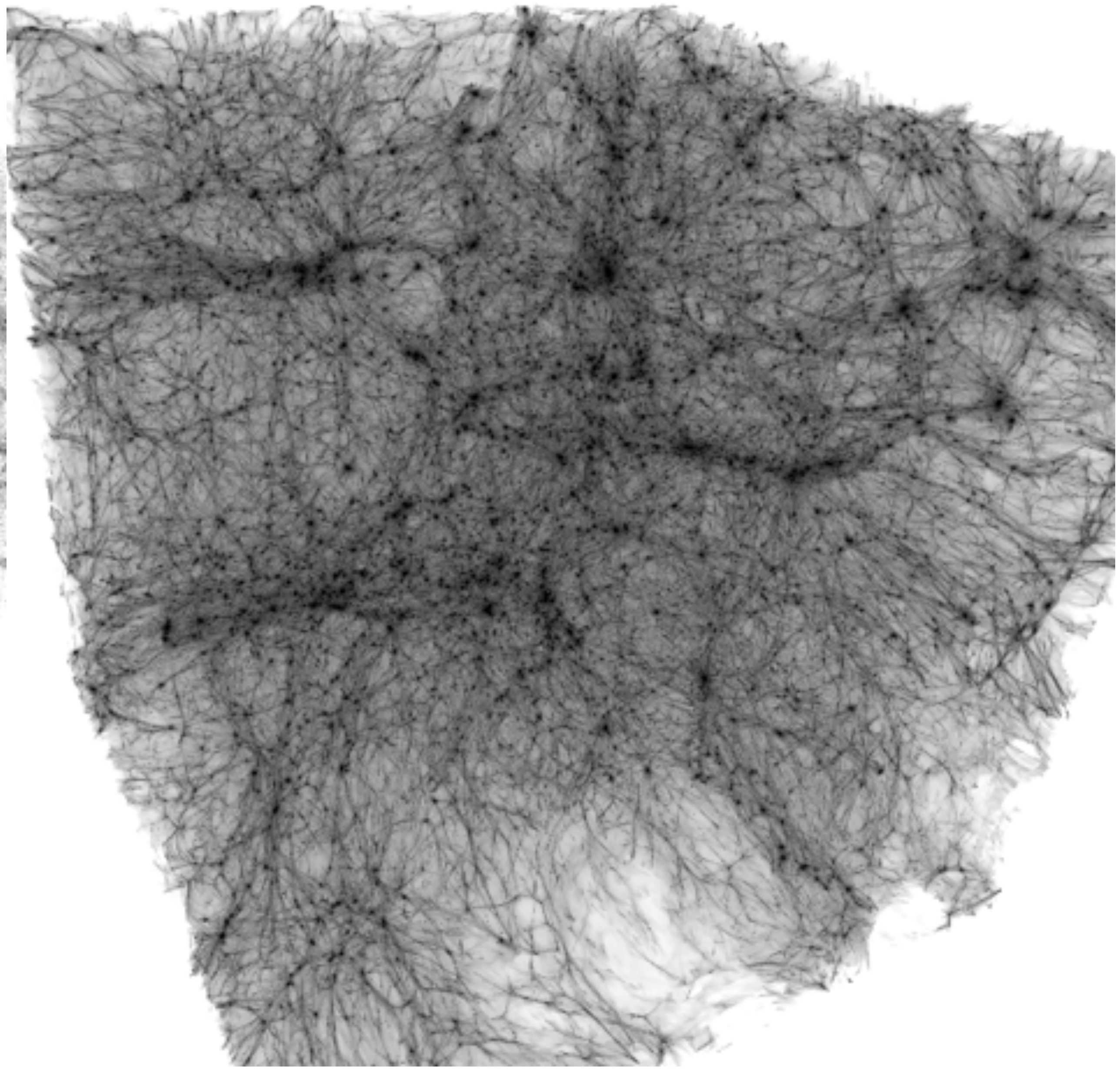


analysis

Three dimensions



rendering points for particles.



rendering tetrahedral phase space cells.

Same simulation data! (Abel, Hahn, Kaehler 2012)

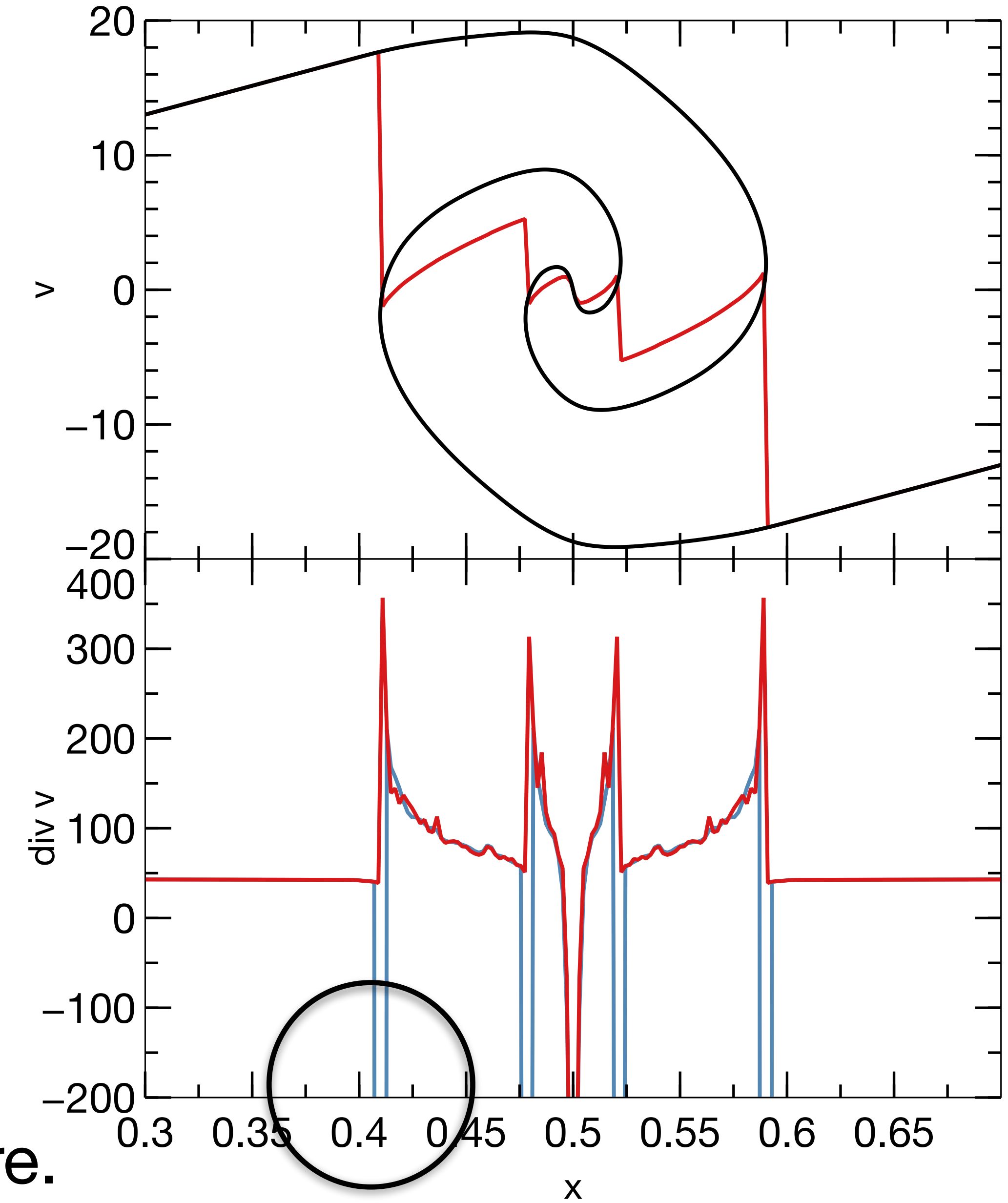
Derivatives of the bulk velocity field

- **Discontinuities make ordinary derivatives ill-defined without coarse-graining!**
- **Away from discontinuities:**
  Need to explicitly evaluate action of derivative on **projected** field:

$$\nabla \cdot \langle \mathbf{v} \rangle = \left\langle (\nabla \log \rho) \cdot (\mathbf{v} - \langle \mathbf{v} \rangle) \right\rangle + \langle \nabla \cdot \mathbf{v} \rangle$$

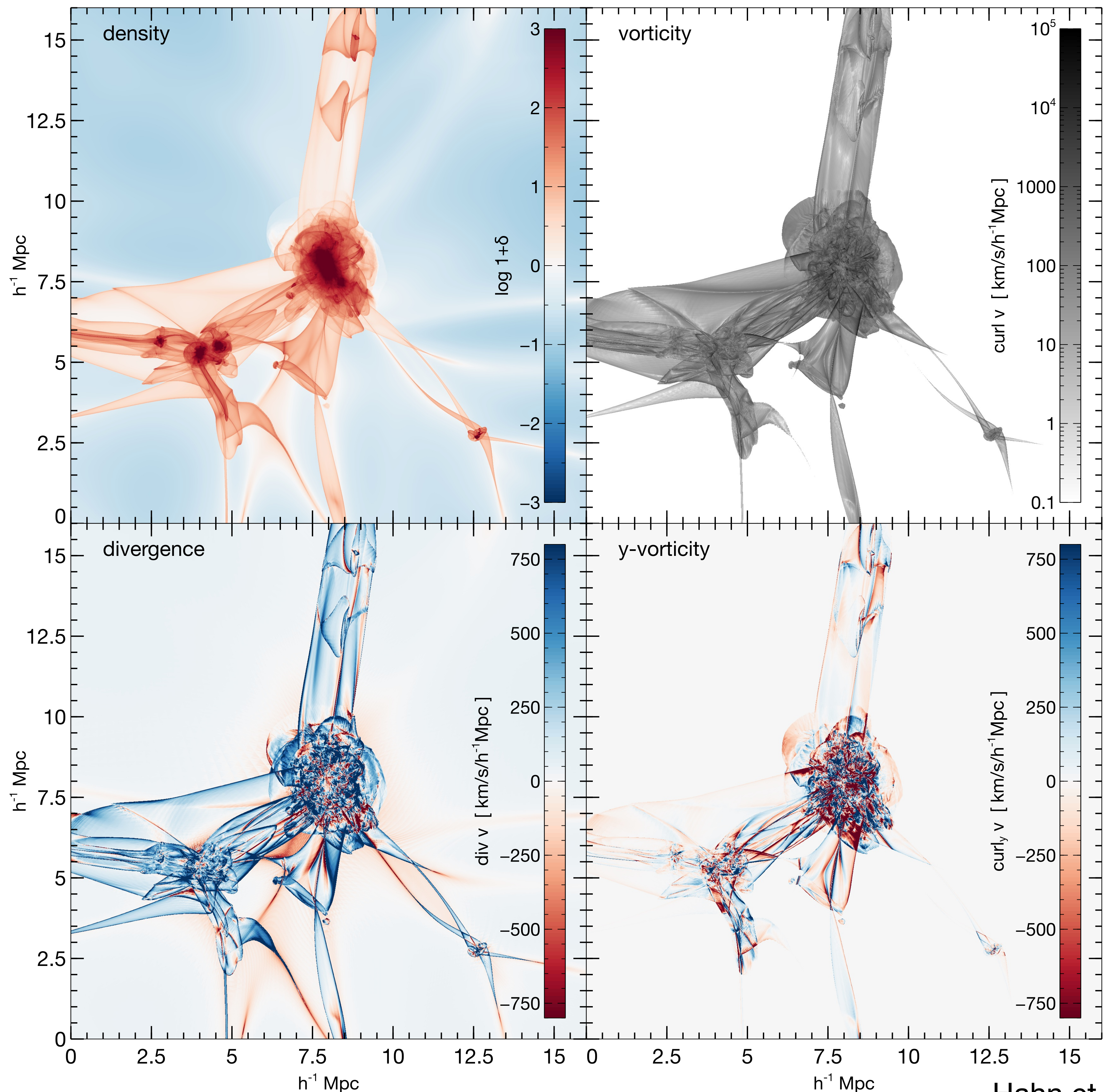
$$\nabla \times \langle \mathbf{v} \rangle = \left\langle (\nabla \log \rho) \times (\mathbf{v} - \langle \mathbf{v} \rangle) \right\rangle + \langle \nabla \times \mathbf{v} \rangle$$

- **Vorticity for std. gravity pure multi-stream phenomenon!!**
- **At discontinuities:**
  Derivatives are singular, but have finite measure.



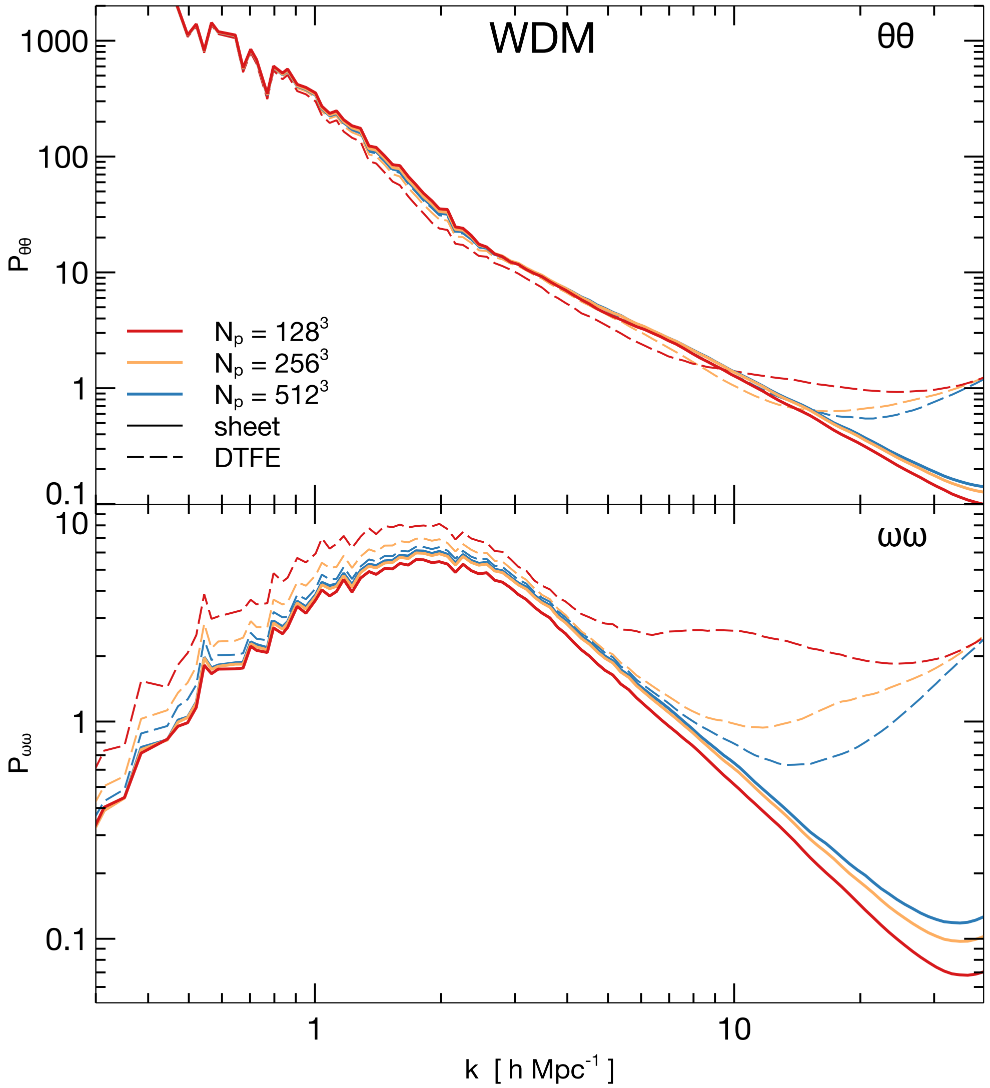
compressive singularities at caustics (=motion of caustics)

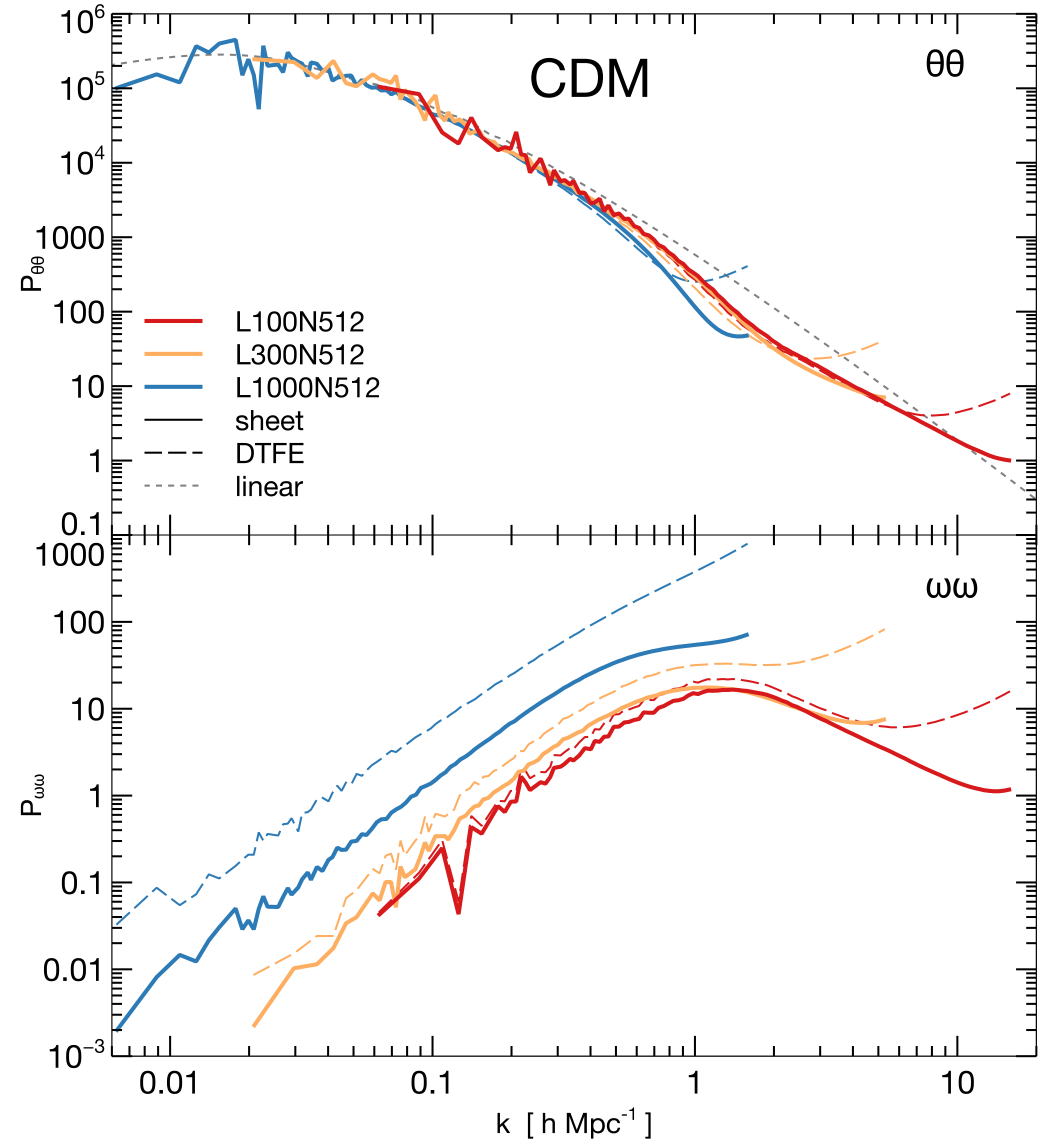
Properties of the cosmic velocity field II

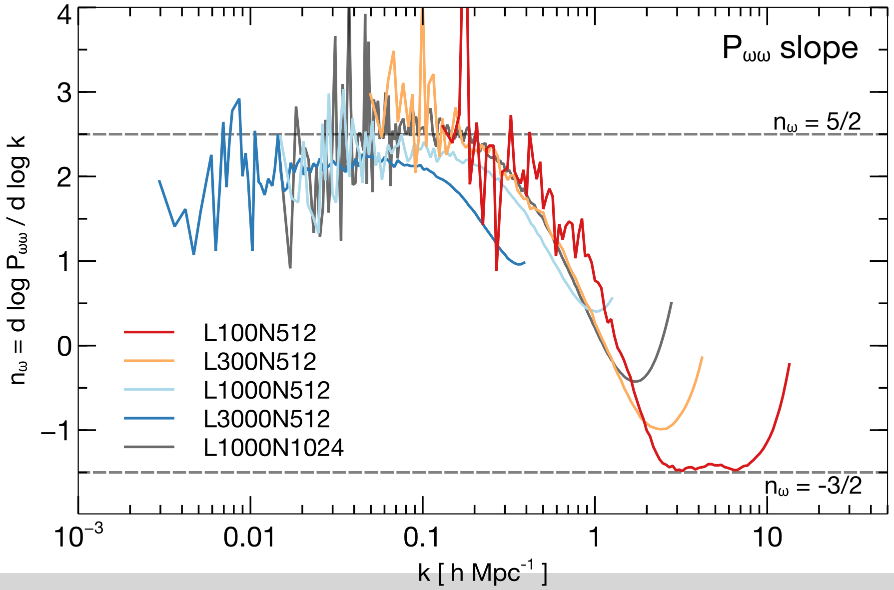


Hahn et al. 2014a

Spectral properties of the cosmic velocity field I







- Faster convergence (for WDM: convergence!)
- Better small scale properties

simulations

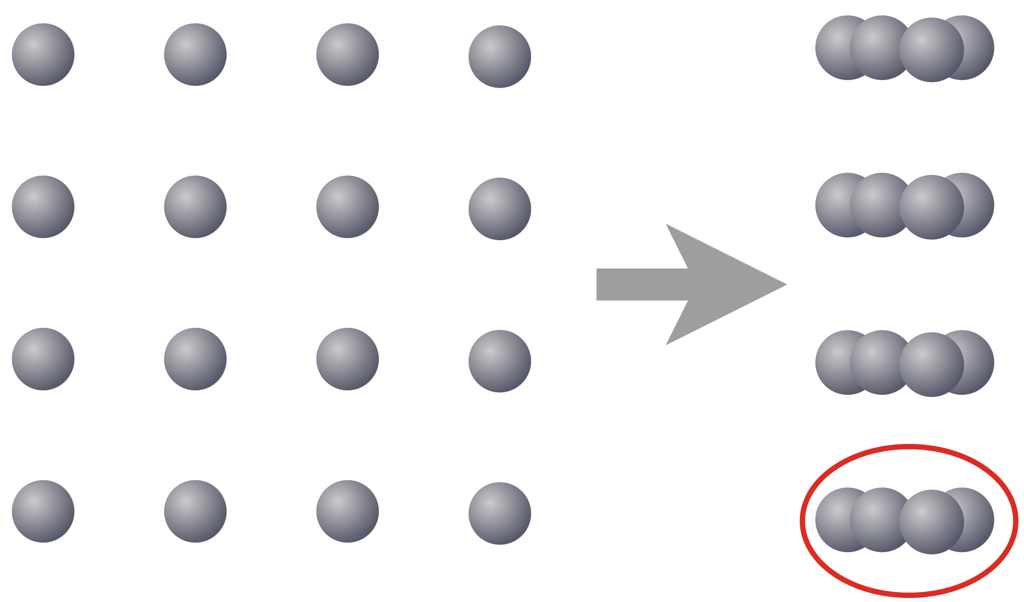
Describing the density field & softening II

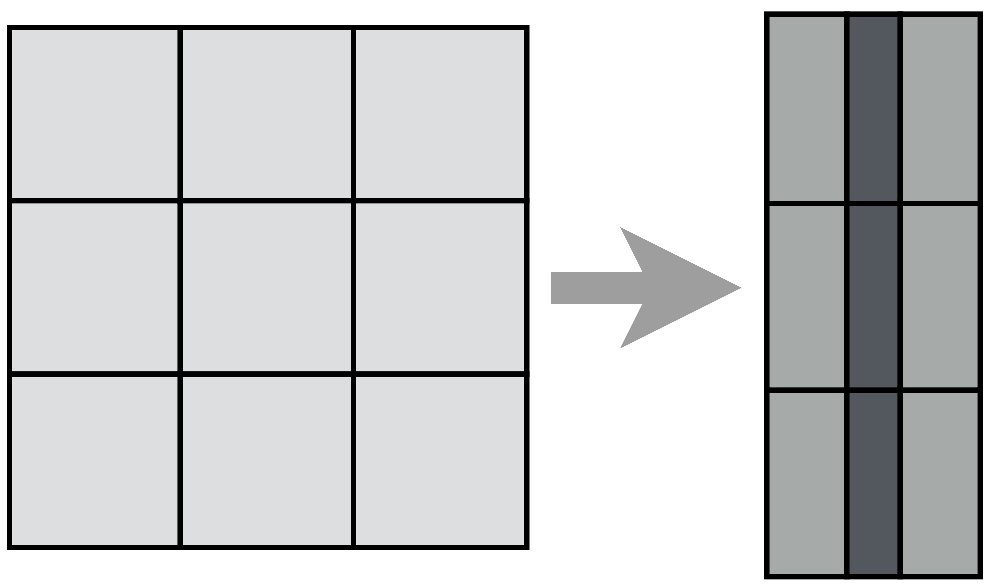
$$\rho = m_p \sum \delta_D(x - x_i) \otimes W$$

need softening,
no knowledge what it
should be (empirical?)

$$\rho = m_p \sum_{\text{streams}} \left| \det \frac{\partial x_i}{\partial q_j} \right|^{-1}$$

self-adaptive

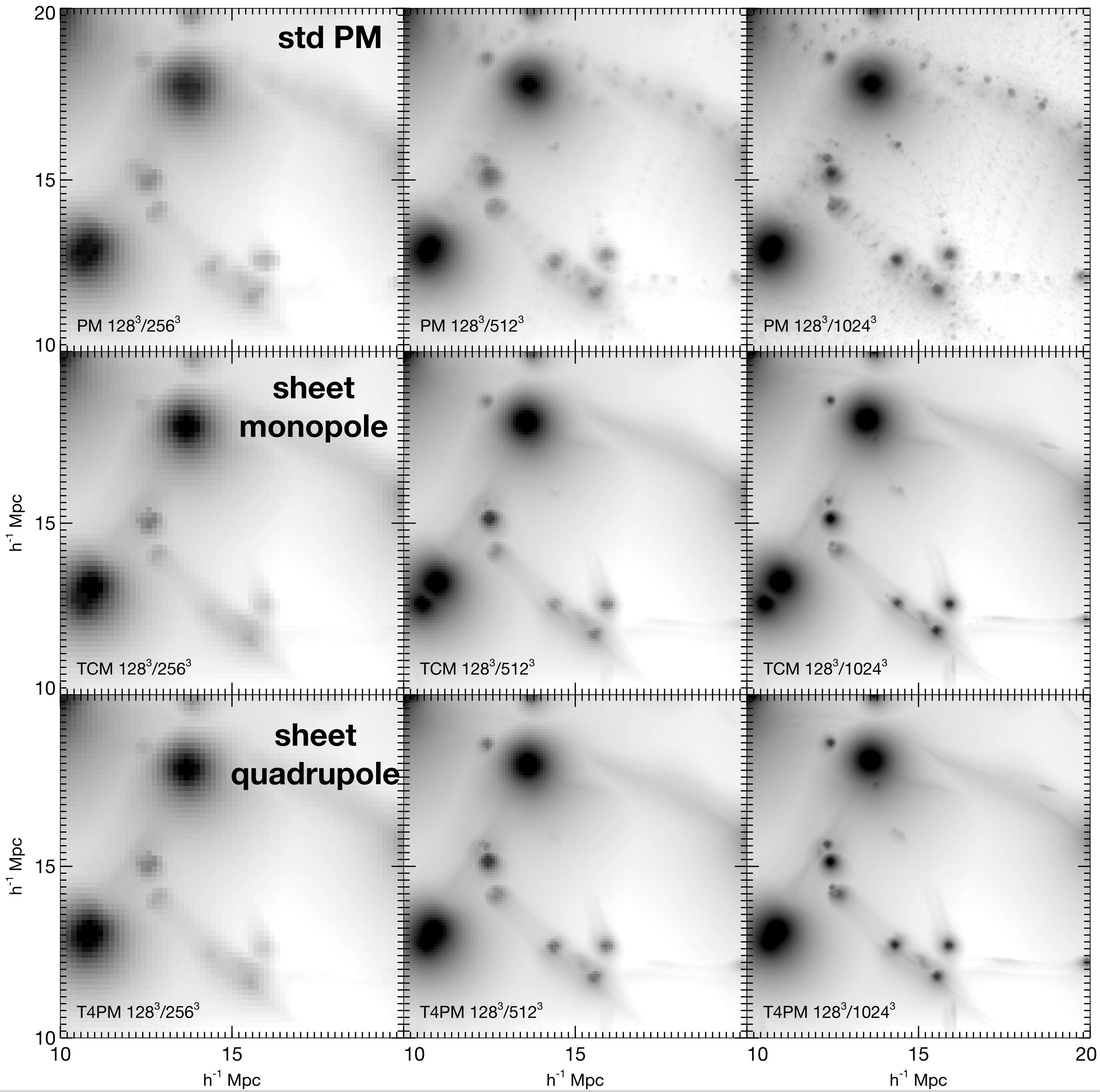




what are the evolution equations for W?
= evolution of the local manifold!

300eV toy WDM problem

fixed mass resolution, varying force resolution:



force res.

features become sharper
fragmentation appears

sheet tesselation
based method cures
artificial fragmentation

but halos
become too dense!

refinement + higher order!

hi-res N-body

tesselated cube orbiting
in non-harmonic potential

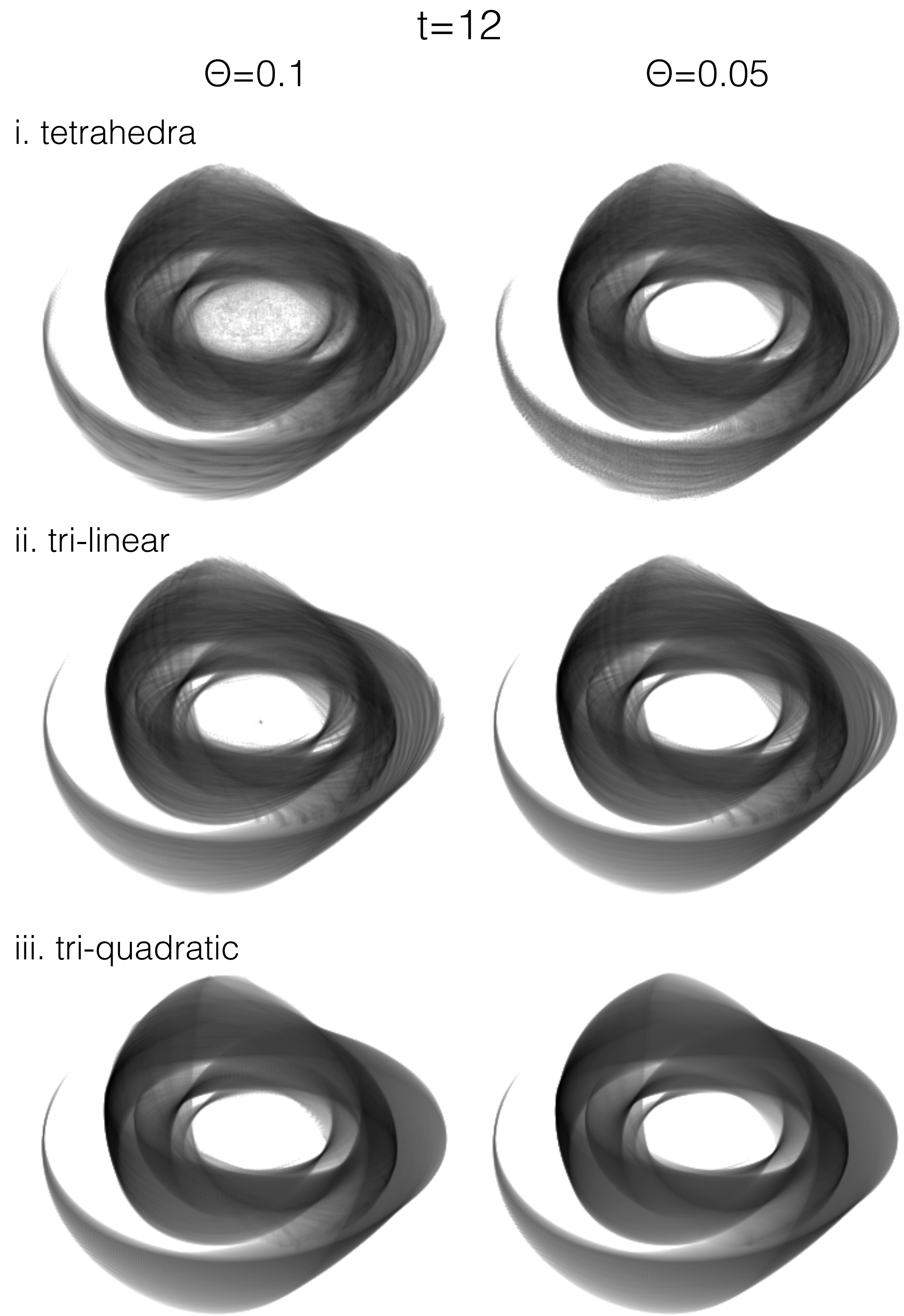
adaptively refined tri-quadratic
phase-space element

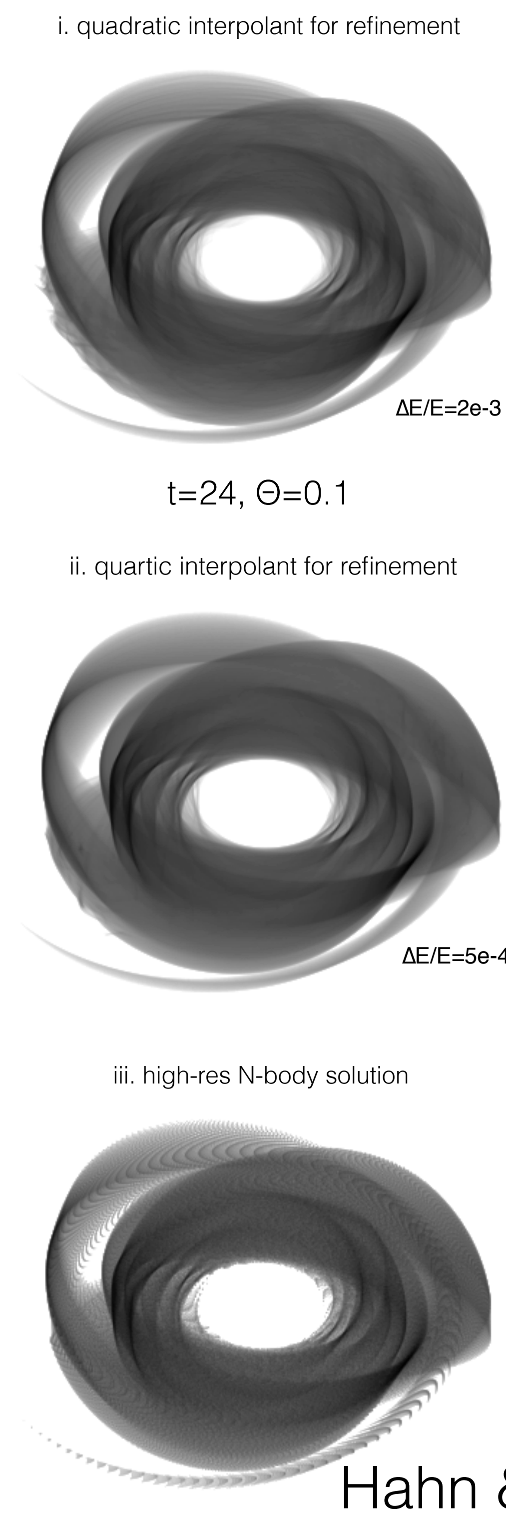
first alternative to N-body in highly non-linear regime!

+ able to track fine-grained phase space

Hahn & Angulo 2015

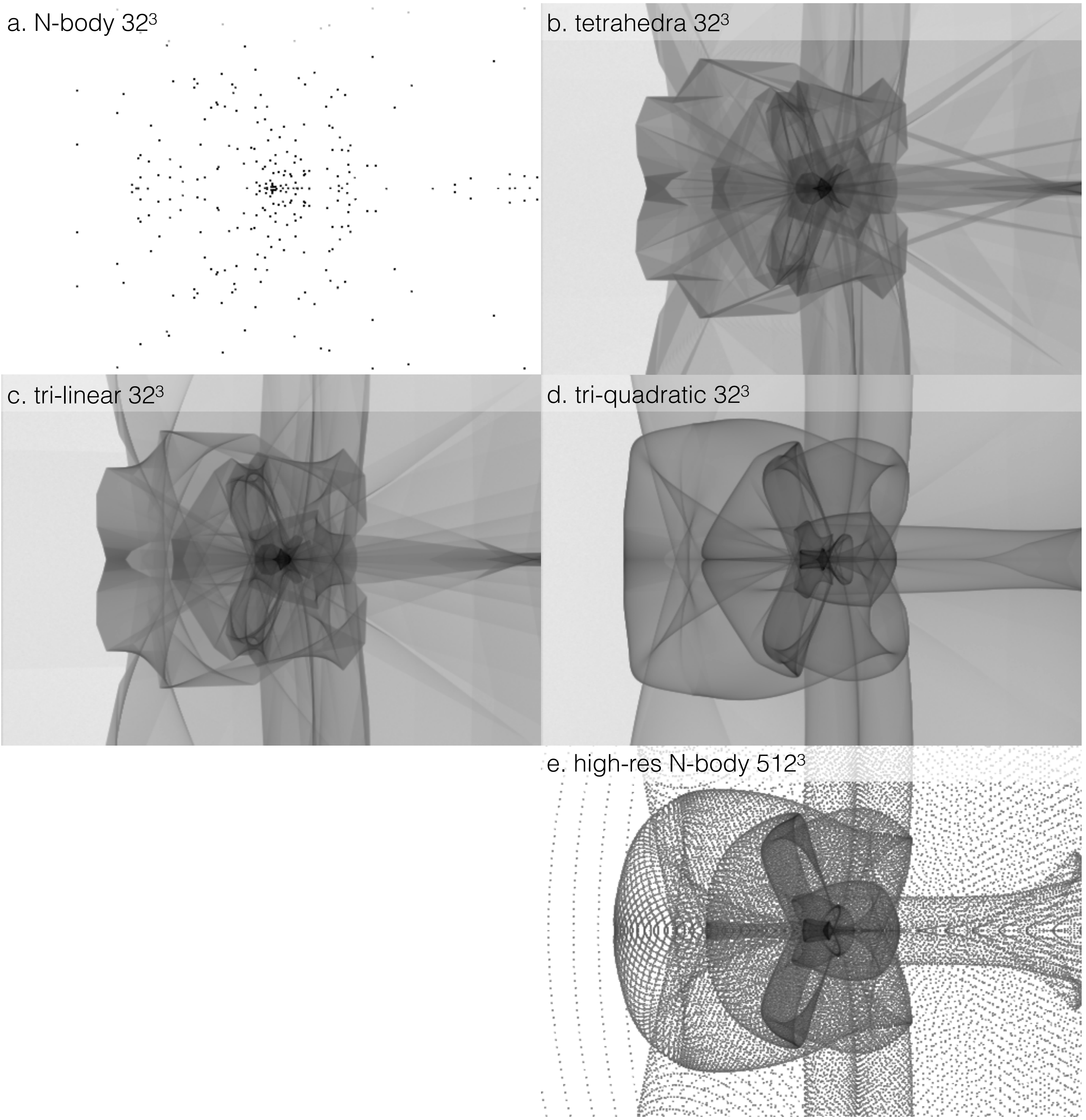
Final results with refinement

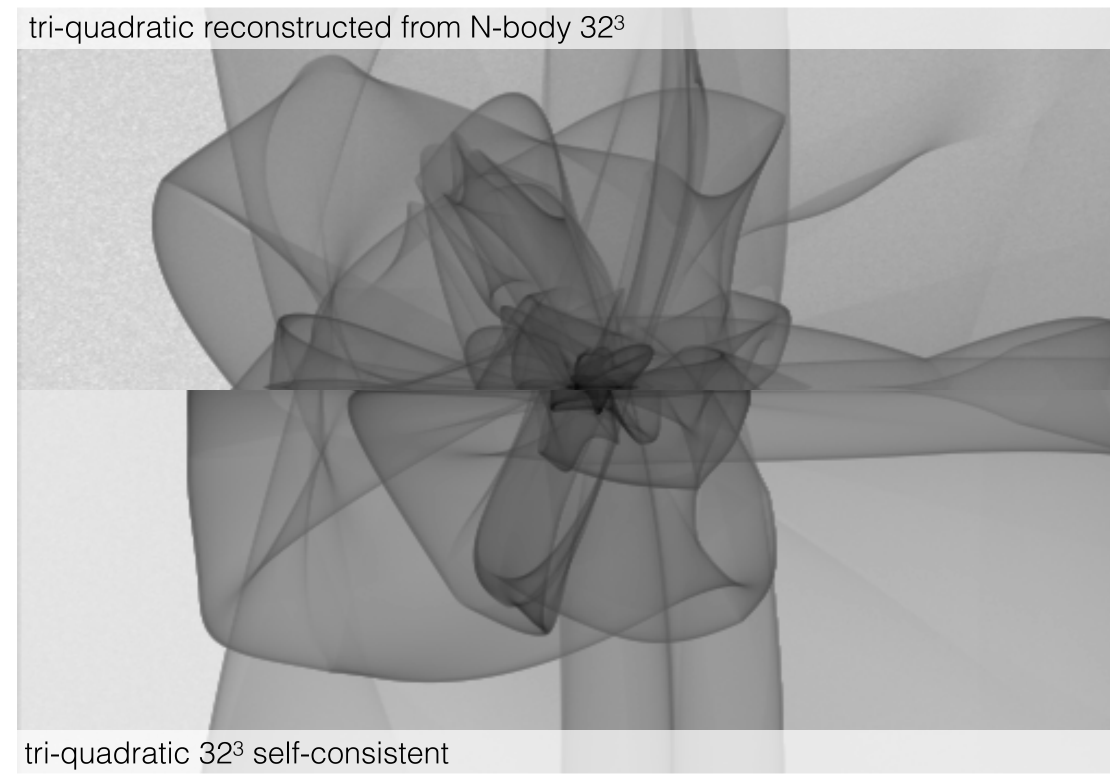




Hahn & Angulo 2015

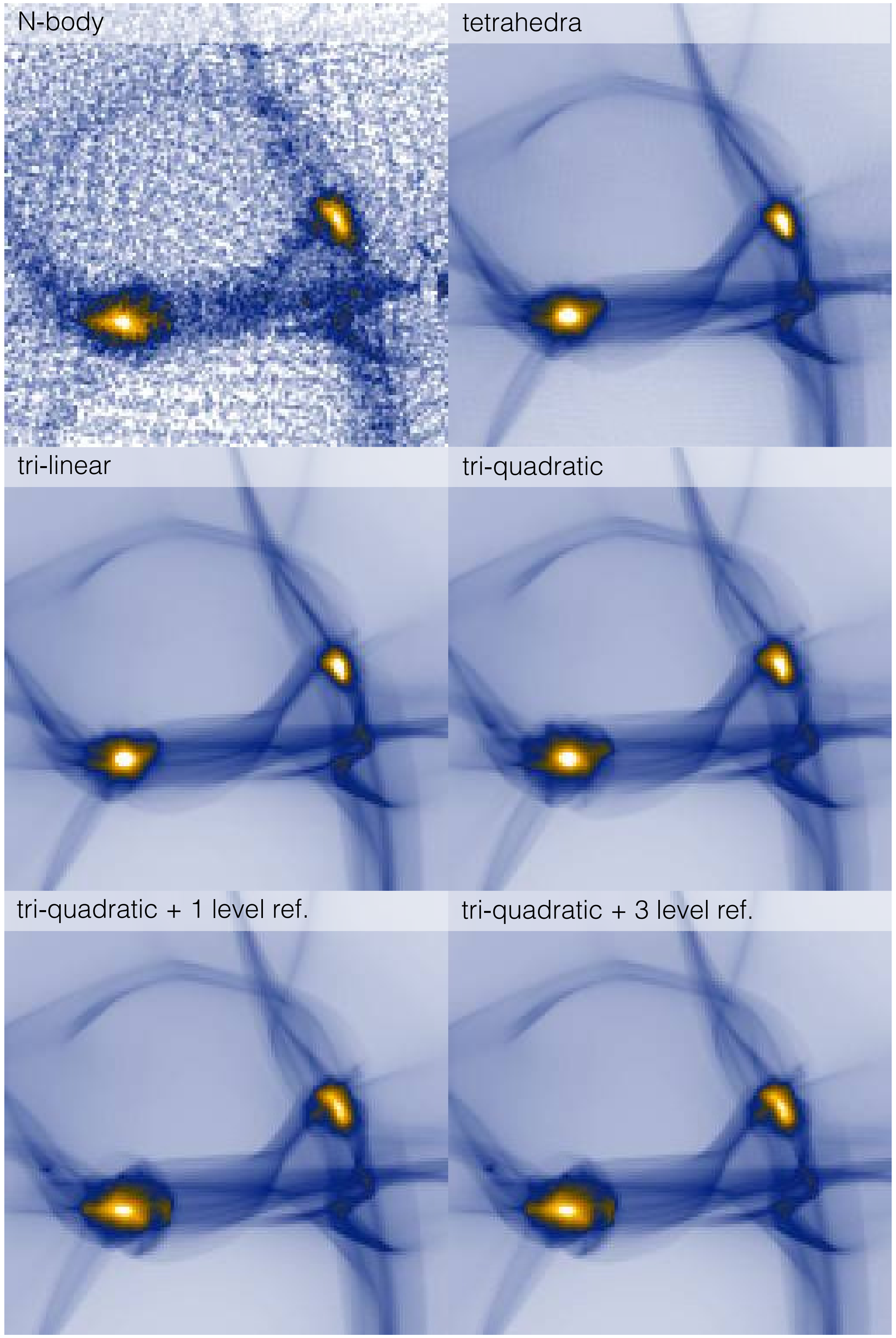
How noisy are N-body sims?

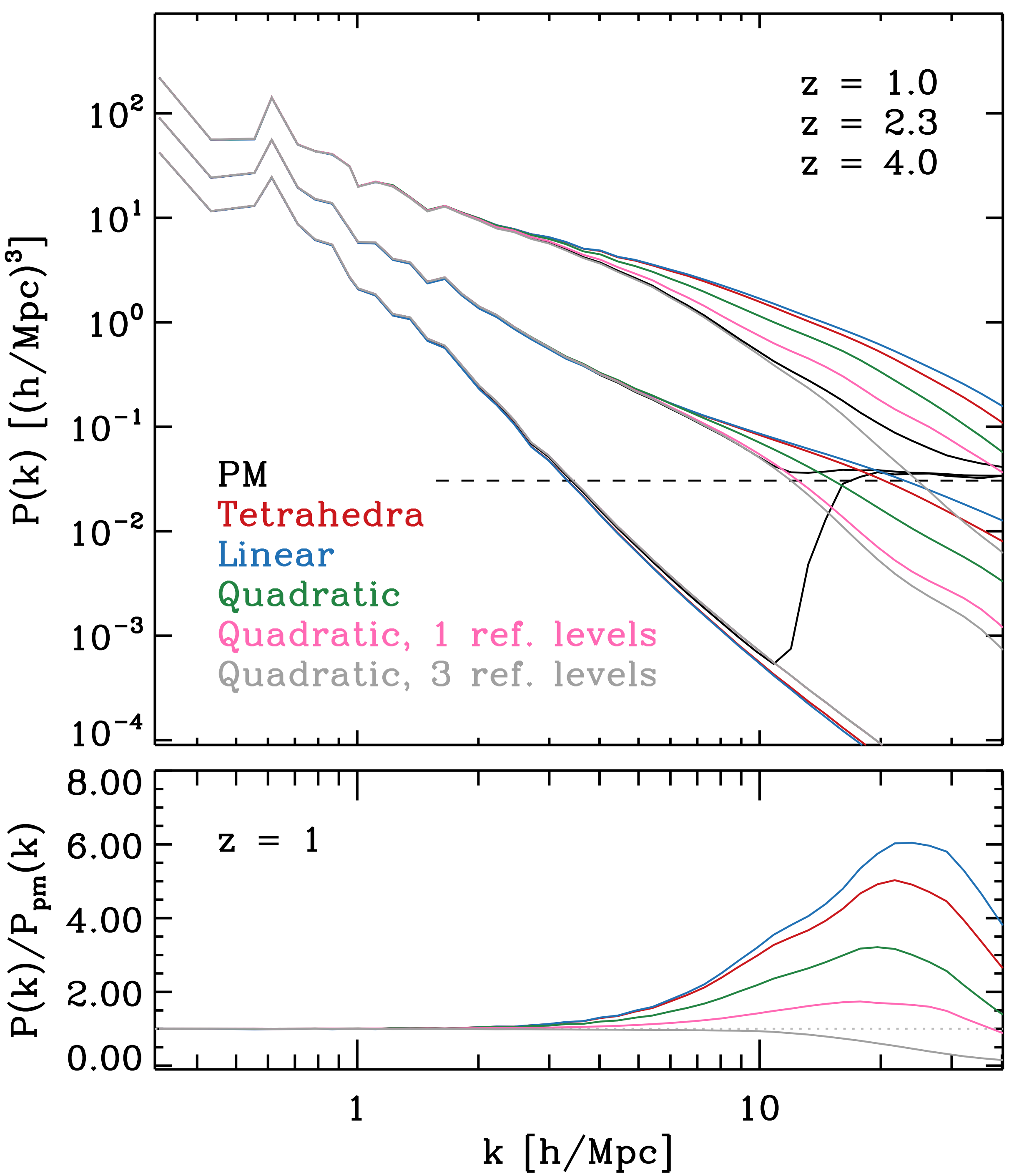




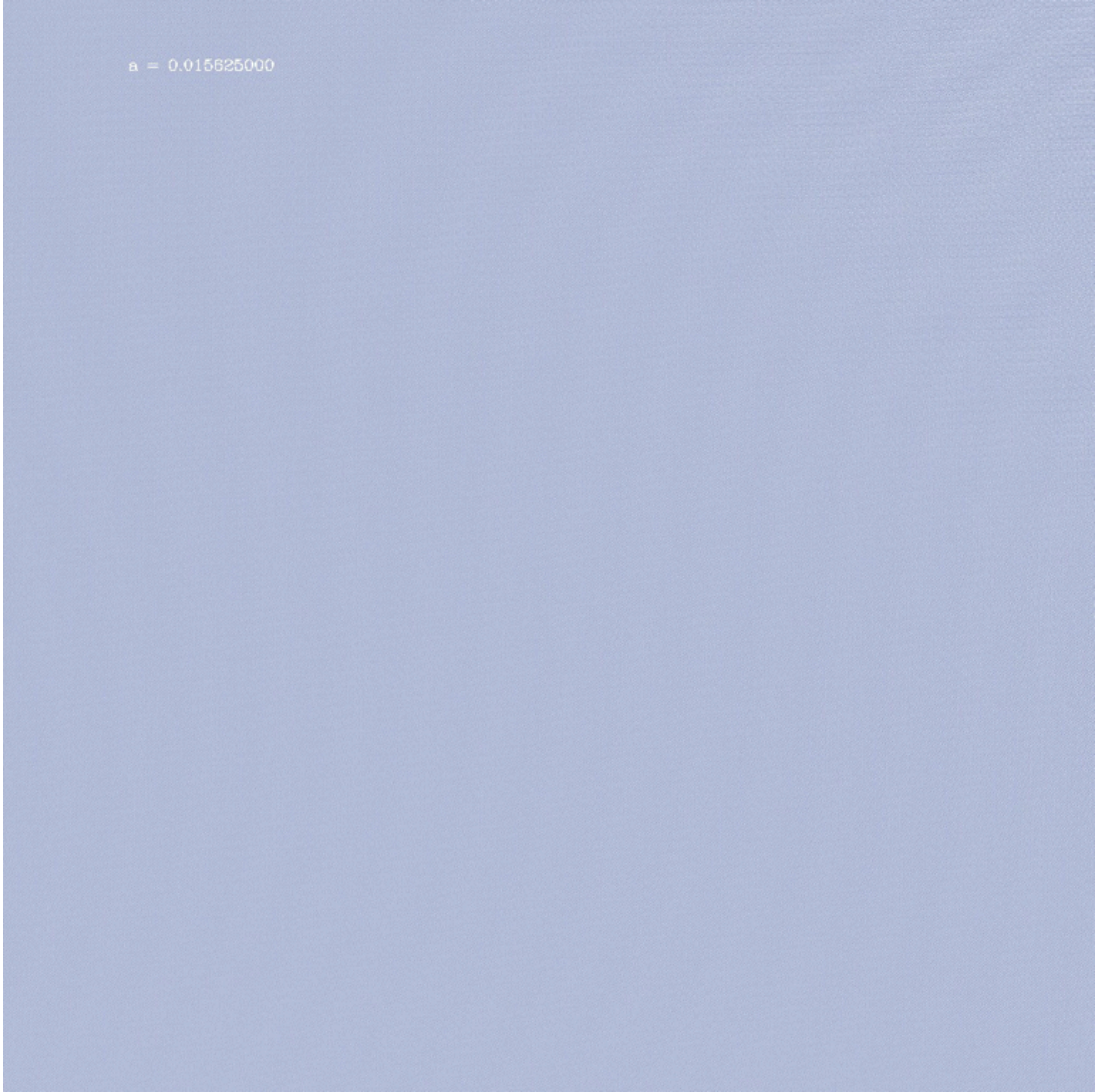
Hahn & Angulo 2015

cosmological simulations w/ refinement

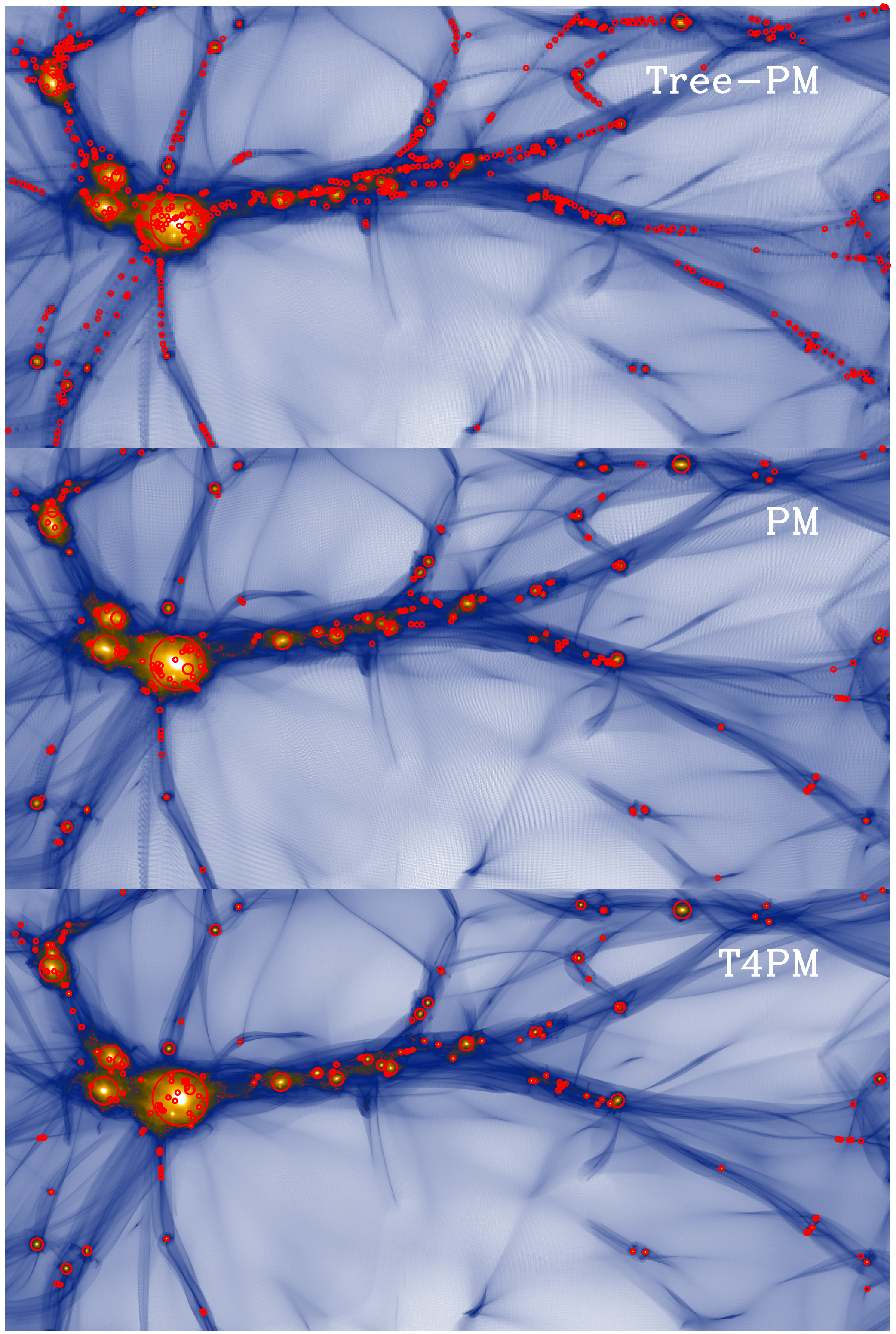


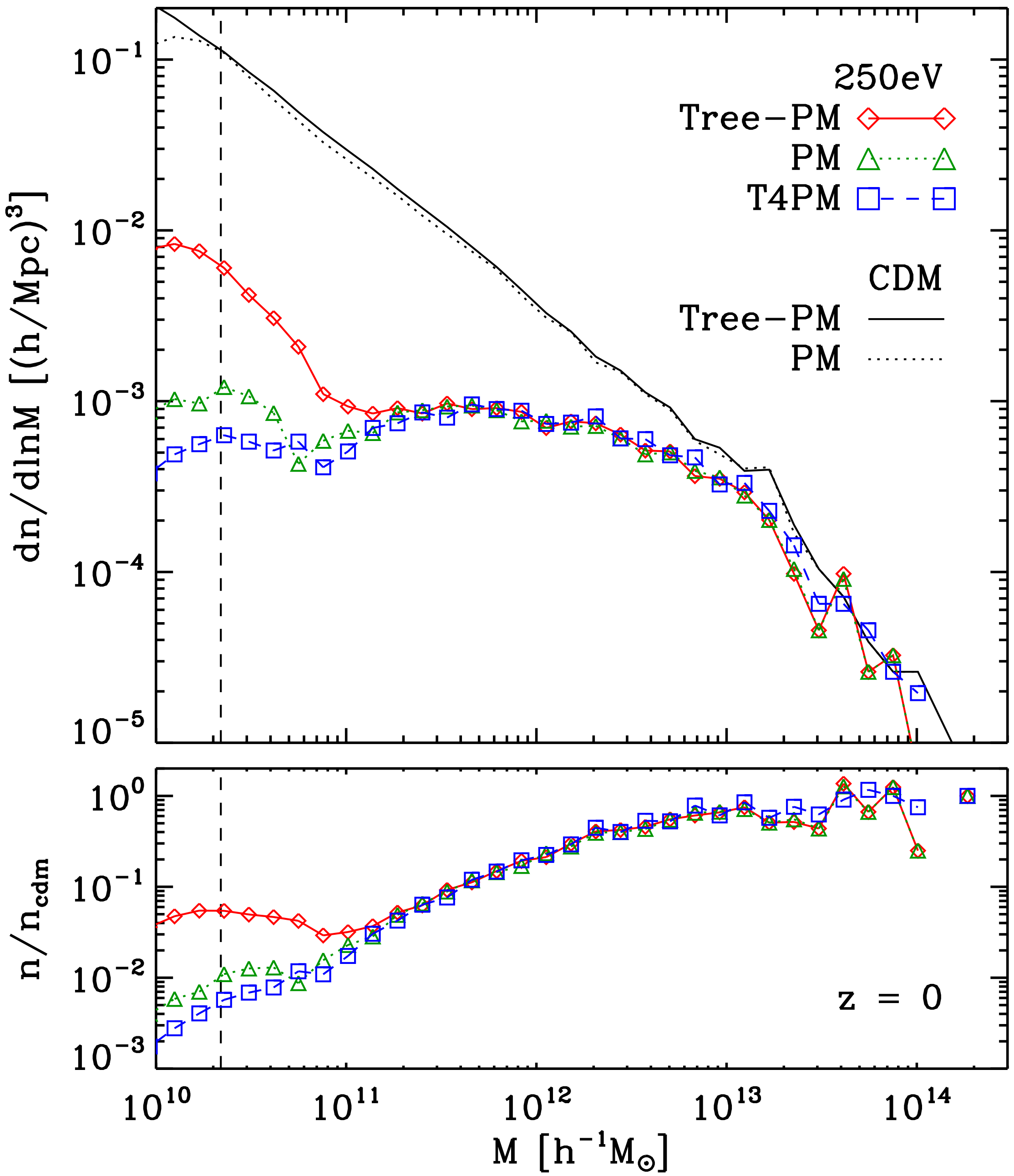


Hahn & Angulo 2015


a = 0.015625000

First determination of WDM halo mass function!



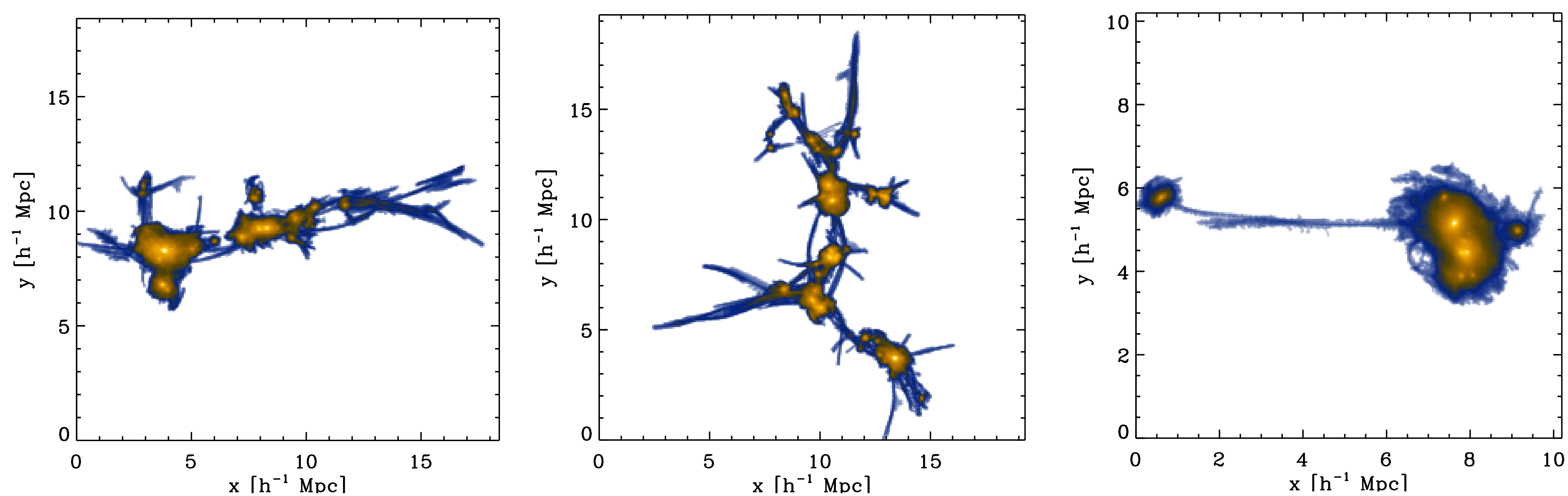


Angulo, Hahn & Abel 2013

Towards the WDM mass function...

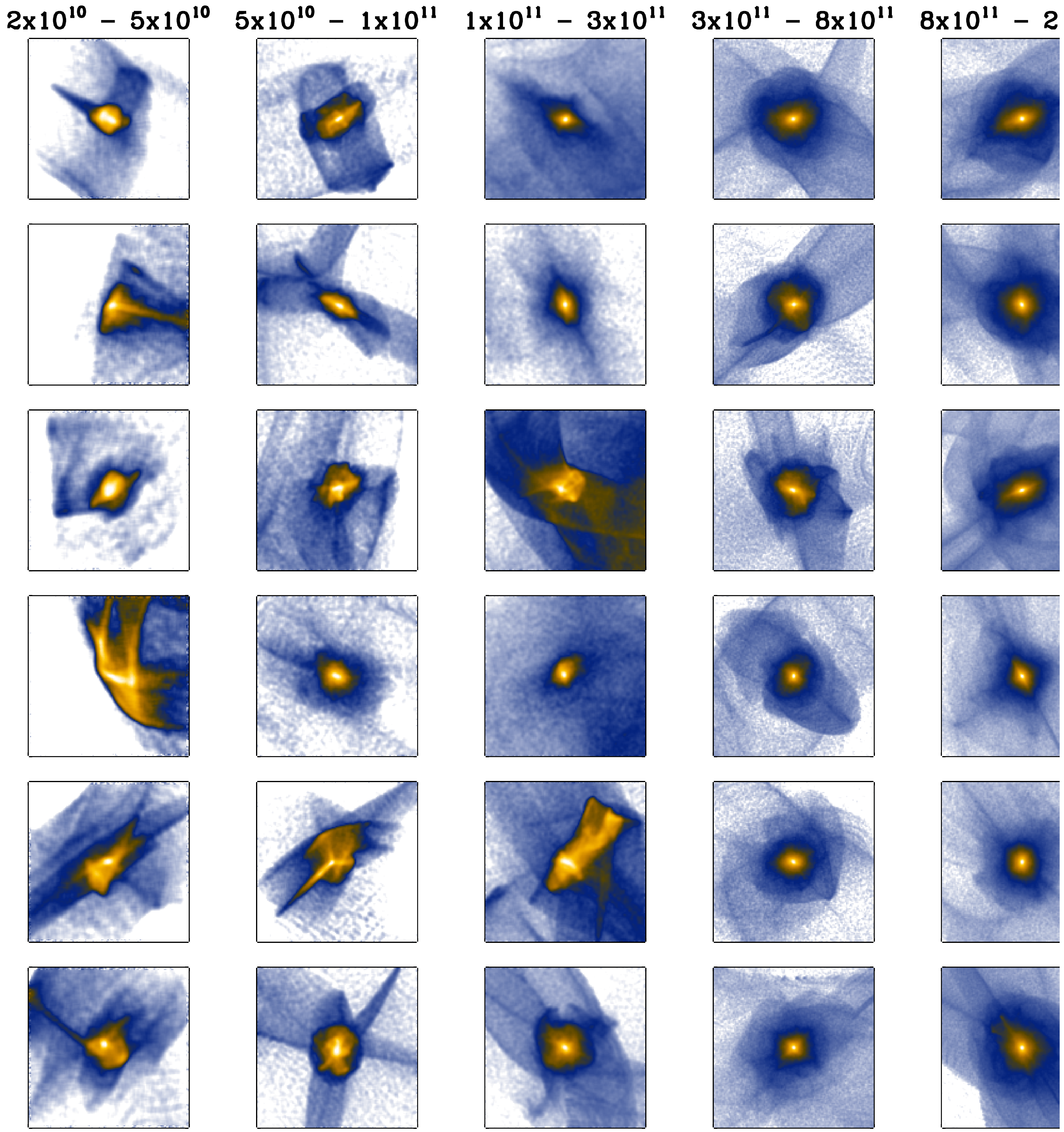
...halo finding becomes challenging

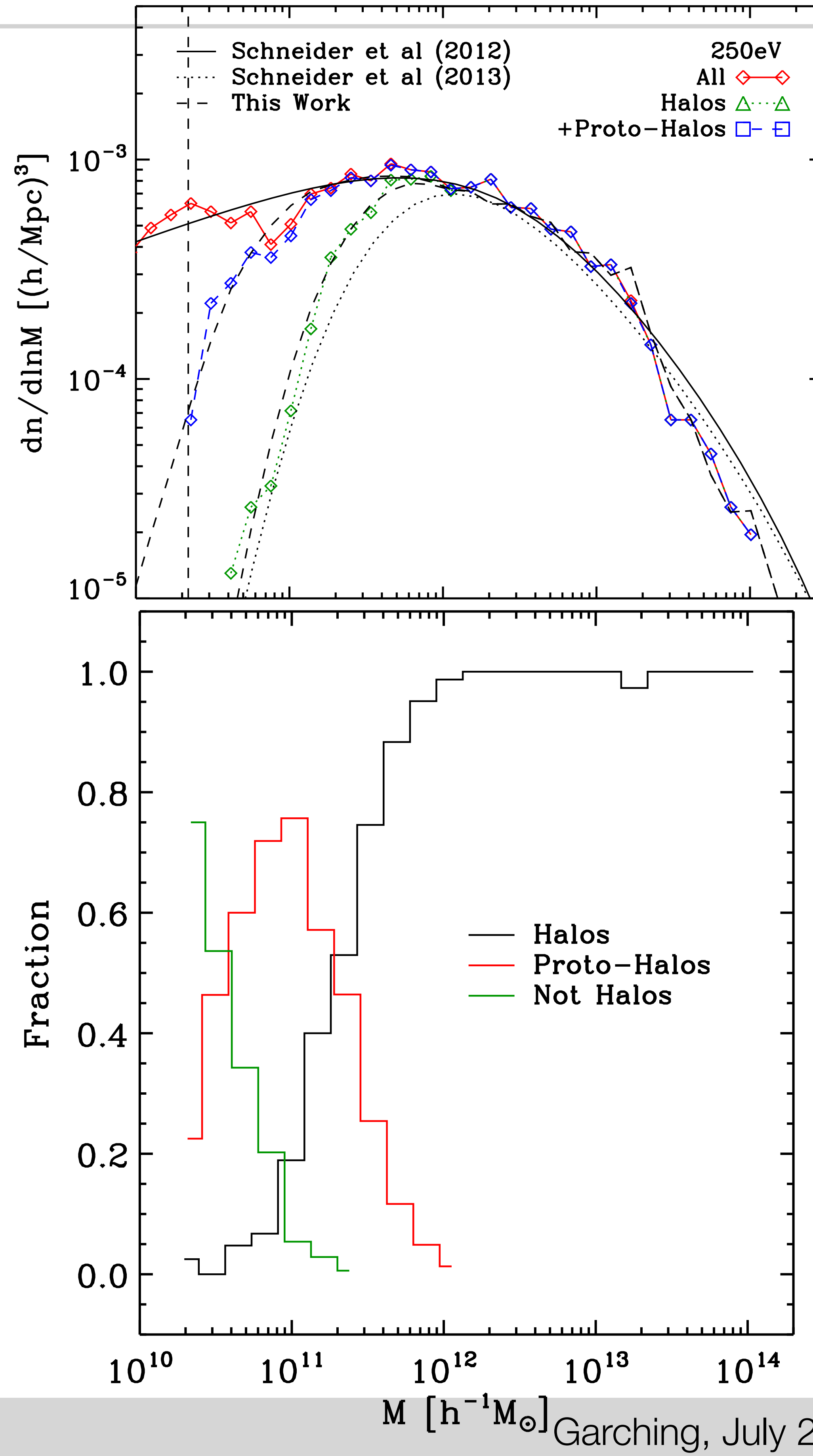
Very dense cores of filaments, linking the halo structures



More work has to be done to understand structure formation.
what do baryons do in such a universe? we don't know yet!

Structures at different masses...





Are at different stages of formation...

Conclusions

- Lagrangian elements can give new insights into existing simulations (density/velocity fields, multi-stream analysis,…)
- Provide also self-consistent simulation technique. (functional when using high-order and adaptive refinement)
- Solves fragmentation problems of N-body
- requires refinement to ensure energy conservation
- First methodological test of N-body in deeply non-linear regime
- Stay tuned for halo properties…