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Perturbation Theory and large deviation functions in cosmology

Self gravitating fluids

- *A multi-component formulation*

$$\Psi_a(\mathbf{k},\eta) = \begin{pmatrix} \delta(\mathbf{k},\eta) \\ \theta(\mathbf{k},\eta) \\ \dots \end{pmatrix}$$

- *Dynamical equations (in Fourier space)*

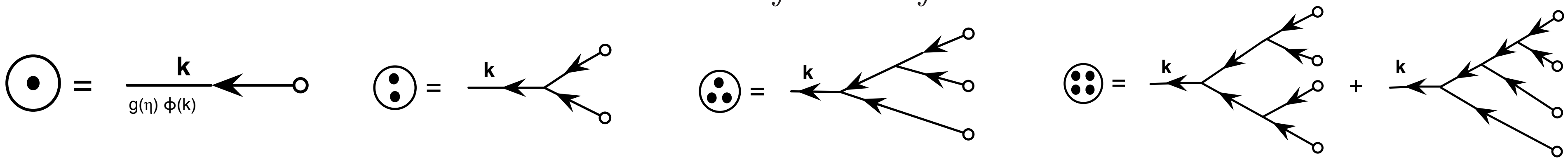
$$\frac{\partial}{\partial\eta}\Psi_a(\mathbf{k},\eta) + \Omega_a{}^b(\eta)\Psi_b(\mathbf{k},\eta) \quad = \quad \gamma_a{}^{bc}(\mathbf{k}_1,\mathbf{k}_2)\ \Psi_b(\mathbf{k}_1,\eta)\ \Psi_c(\mathbf{k}_2,\eta)$$

convolution is implicit

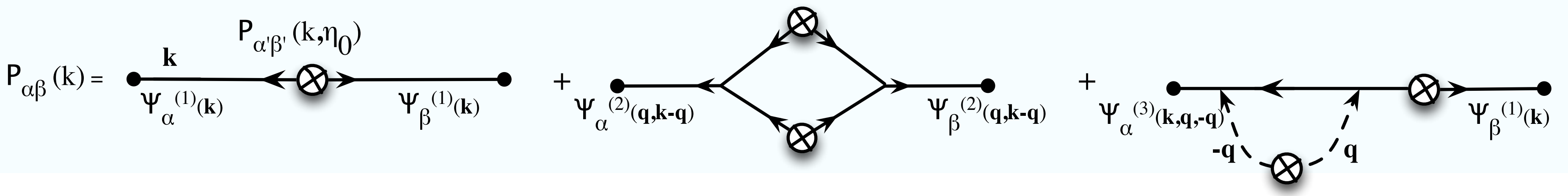
  - *Explicit results will be given here for a single-component pressureless fluid*
  - *detailed effects of baryons versus DM can be taken into account (Somogyi & Smith 2010; FB, Van de Rijt, Vernizzi '12) with a 4-component multiplet*
  - *Same structure also for non-interacting relativistic particles (neutrinos) with multiple flow description (Dupuy and FB, '14, '15)*

- Diagrammatic representation

$$\delta_n(\mathbf{k}) = \int \mathrm{d}^3\mathbf{q}_1 \dots \int \mathrm{d}^3\mathbf{q}_n\, \delta_D(\mathbf{k}-\mathbf{q}_{1\dots n}) F_n(\mathbf{q}_1,\dots,\mathbf{q}_n)\delta_1(\mathbf{q}_1)\dots\delta_1(\mathbf{q}_n)$$



- Ensemble averages by glueing diagrams together



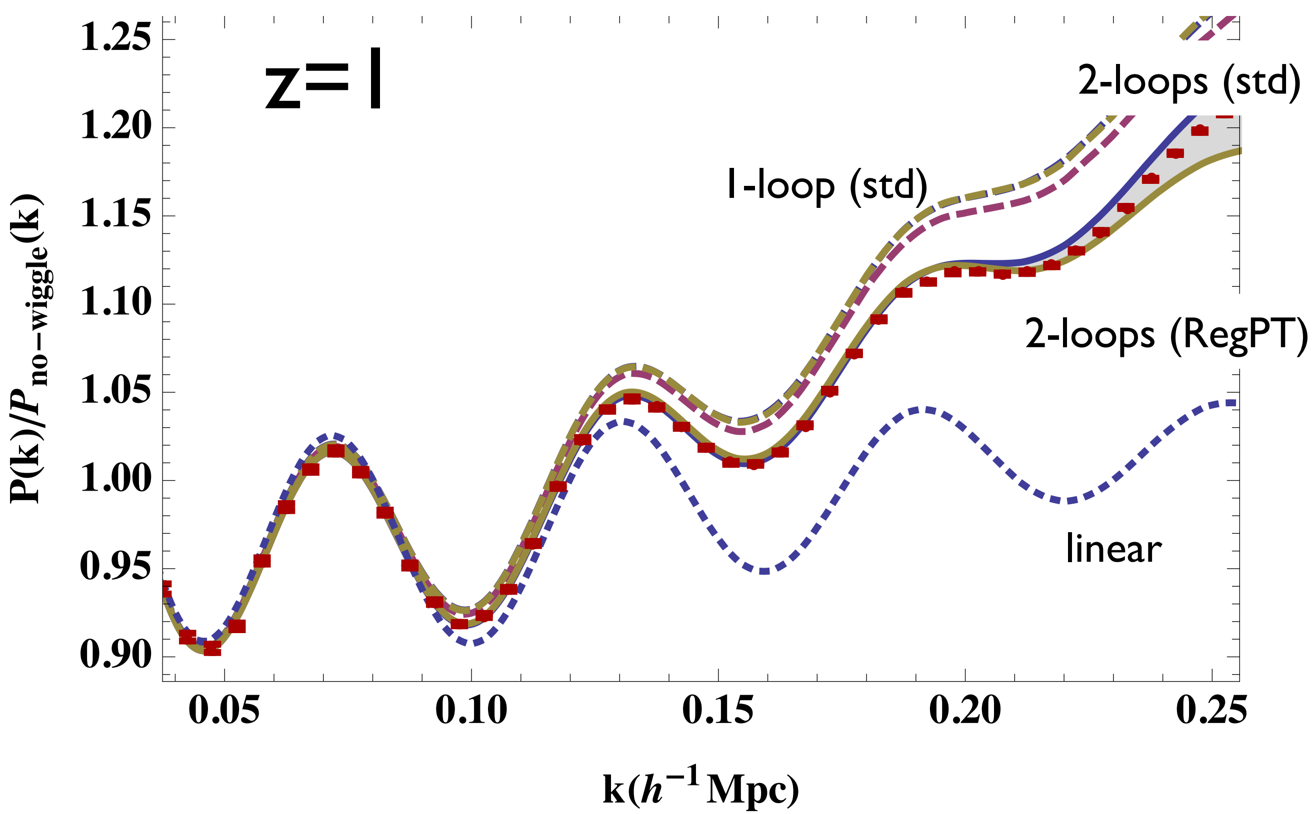
Charting PT

number of loops in standard PT for Gaussian Initial Conditions →

Order of observable in field expansion ↓

	Tree order LO	1-loop NLO	2-loops NNLO	3-loops	...p-loops
2-point statistics	OK	OK	OK	partial exact results	partial resum
3-point statistics	OK	OK (but not systematics)			partial resummations
4-point statistics	OK	OK (but not systematics)			
N-point statistics	OK, in specific geometries (counts in cells)				

- *Not a single way of doing PT calculations*
  - *change of variables or fields : most dramatic is Eulerian to Lagrangian*
  - *re-organisation(s) of the perturbation series (for instance with multipoint propagators introduced in FB, Crocce, Scoccimarro, PRD, 2008)*
  - *PT can then come in many different flavors : SPT, RPT, TRG, RegPT, gRPT, MPT*
- *Power spectra up to 1-loop and 2-loop order*



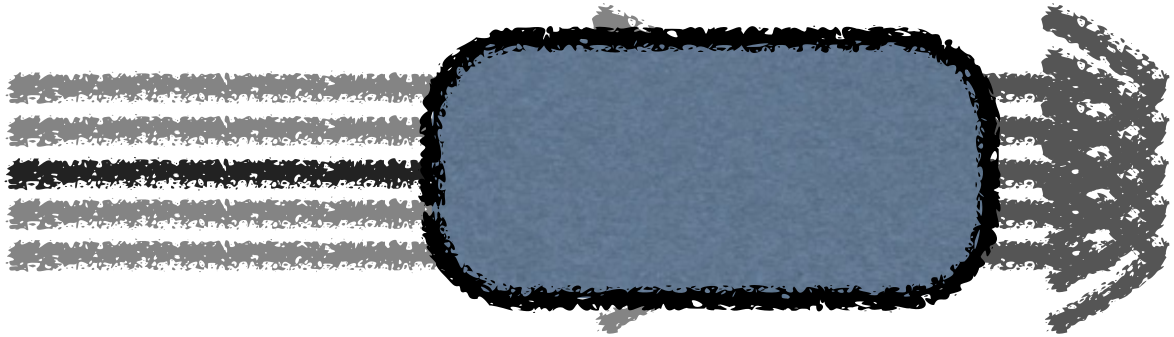
- Public codes are available

RegPT from Taruya , FB, Nishimichi, Codis '12

How far can we go ?

An alternative to the power spectra : response functions

$\delta P^{\mathrm{lin}}(q)$



$\delta P^{\mathrm{nl}}(k)$

$$\mathcal{R}_{\mathcal{M}_1}(k,q) = q\ \frac{\delta P^{\mathrm{nl}}_{\mathcal{M}_1}(k)}{\delta P^{\mathrm{lin}}_{\mathcal{M}_1}(q)}$$

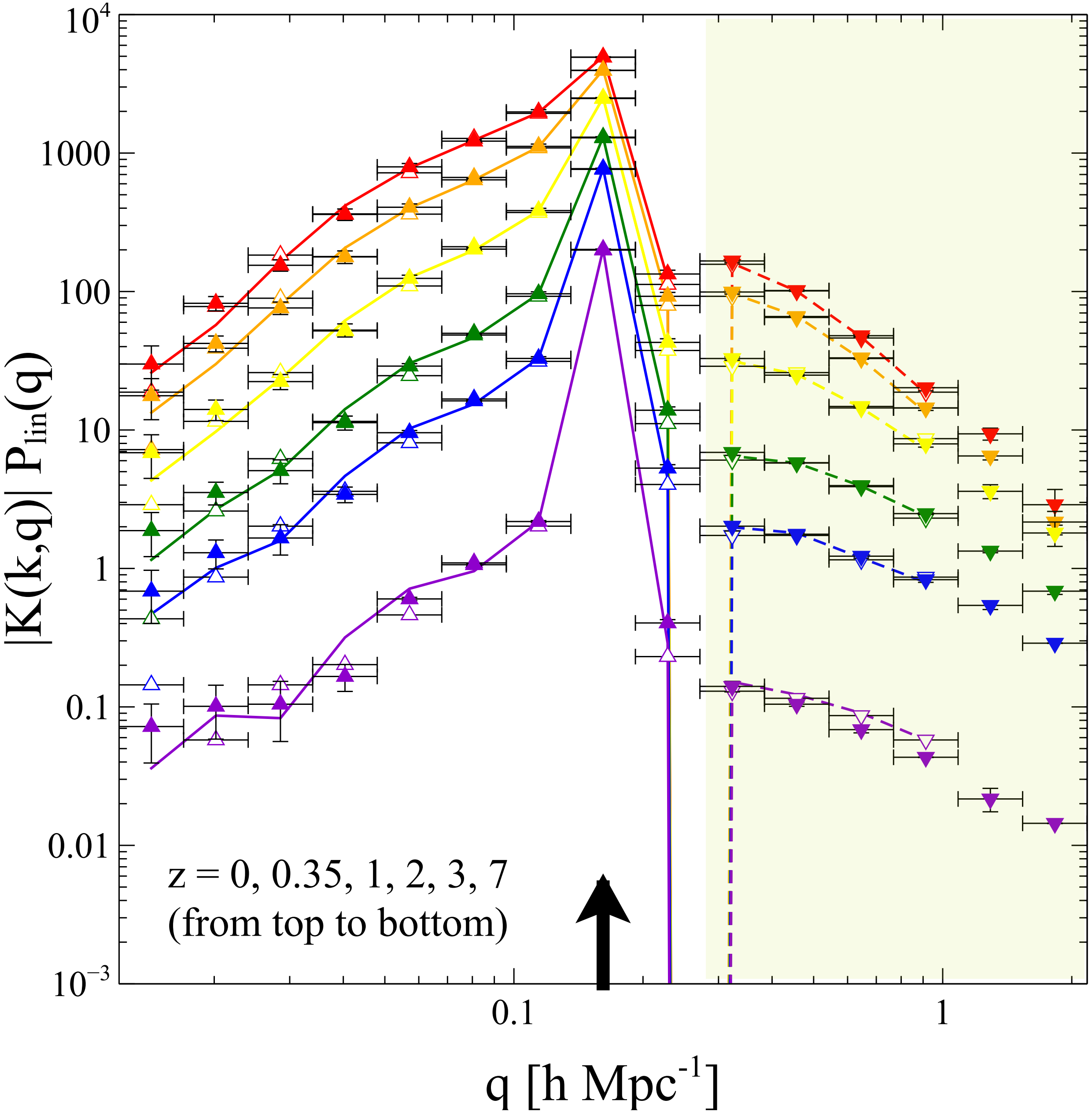
Of direct interest from P(k) predictions:

$$P^{\mathrm{nl}}_{\mathcal{M}_2}(k) \approx P^{\mathrm{nl}}_{\mathcal{M}_1}(k) + \int \frac{\mathrm{dq}}{q}\ \mathcal{R}_{\mathcal{M}_1}(k,q)\ \left[P^{\mathrm{lin}}_{\mathcal{M}_2}(q) - P^{\mathrm{lin}}_{\mathcal{M}_1}(q)\right]$$

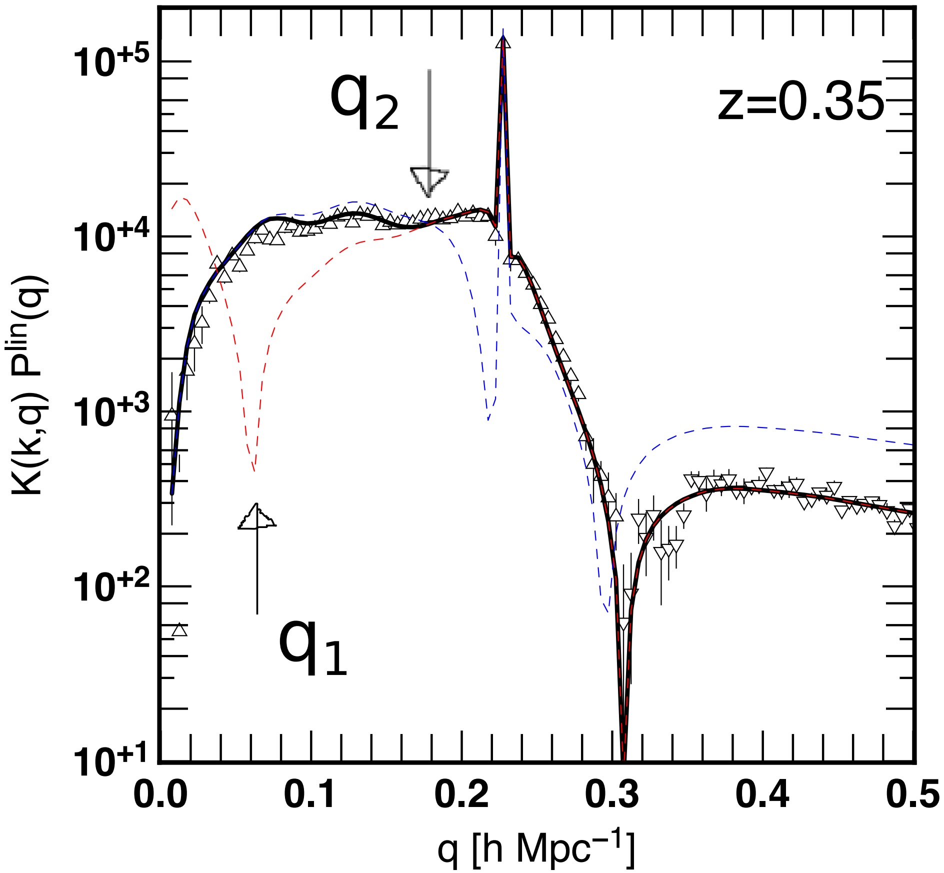
How good can PTs be at predicting response functions ?

first measurement of the response function

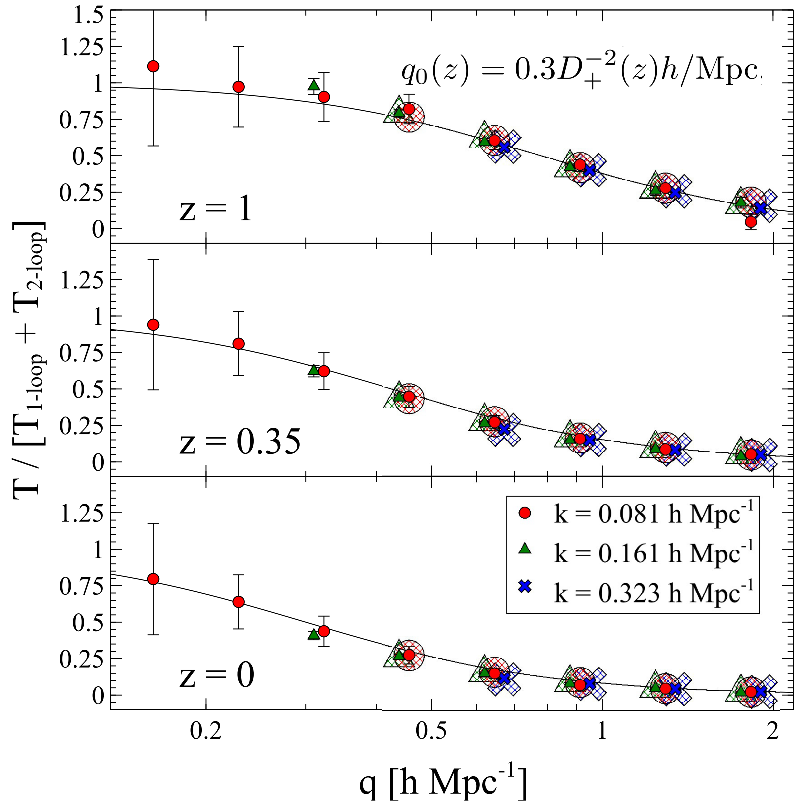
Nishimichi, FB, Taruya, '14



Comparison with 1- and 2-loop results



$$T^{\mathrm{eff.}}(k,q) = \left[T^{1-\mathrm{loop}}(k,q) + T^{2-\mathrm{loop}}(k,q)\right] \frac{1}{1+(q/q_0)^2}$$



- From PT perspective, UV regularization is necessary
- existence of damping is good news (it reduces sensitivity to small scale physics)
- origin is unclear (associated to shell-crossings ?)

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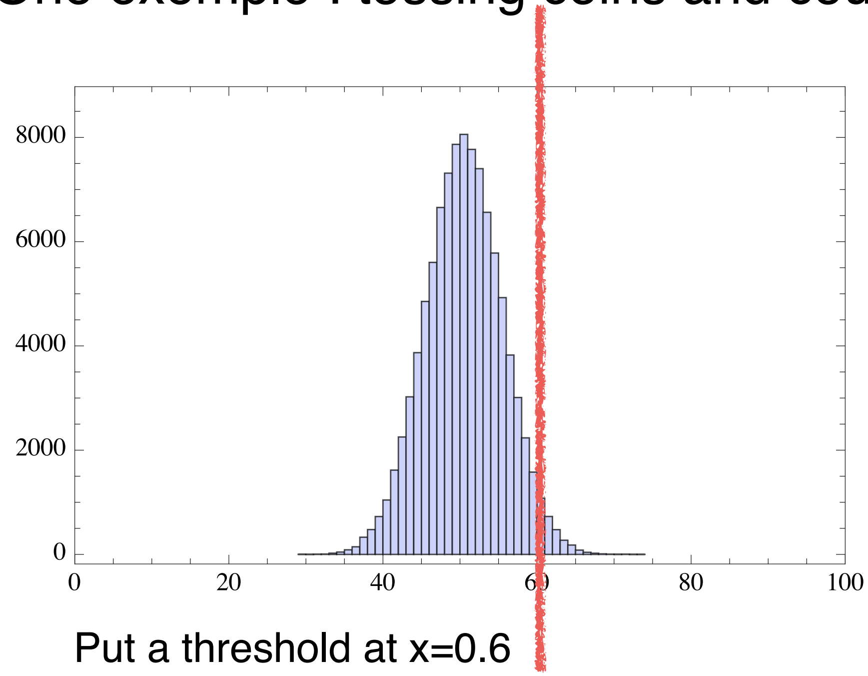
large-deviation regime

Basics of theory of large deviation functions

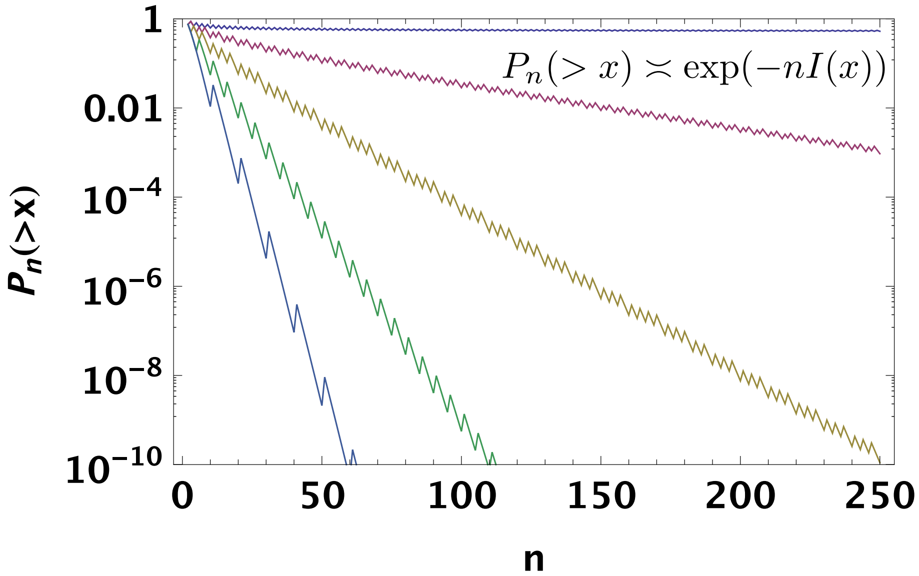
Review paper by Hugo Touchette, '09

Beyond the central limit theorem

One exemple : tossing coins and counting the number of heads $x = \frac{1}{n} \sum_n t_n$



Put a threshold at x=0.6

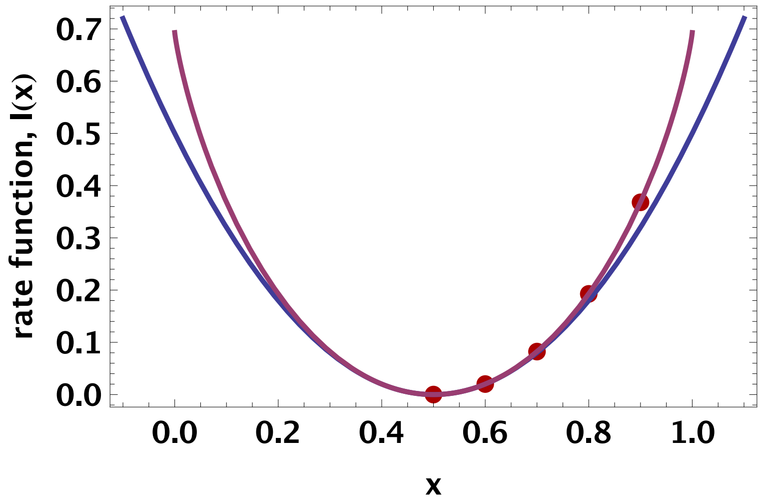


Central limit theorem : $I(x) = 2(x - 0.5)^2$

Exact result : $I(x) = x \log[x] + (1 - x) \log[1 - x] + \log[2]$

The cumulant generating function : $\varphi(\lambda) = \log\left(e^{\lambda}/2 + 1/2\right)$

Cramér's Theorem : both are Legendre transform of one-another



Key theorems: relation between rate function and cumulant generating function

Consider a random variable x such that, $$x = \frac{1}{n}\sum_n v_i \qquad \left(n \equiv \frac{1}{\sigma^2}\right)$$

The (scaled) cumulant generating function of x is defined as,

$$\varphi(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log\left[\langle e^{n\lambda x}\rangle\right] = \log\left[\langle e^{\lambda v}\rangle\right]$$ *(in case vi are IID)*

The ***Gärtner-Ellis Theorem*** *(Cramér's Theorem for IID): the rate function is the Legendre-Fenchel transform of the (scaled) cumulant generating function*

$$I(\rho) = \sup_\lambda [\lambda\rho - \varphi(\lambda)]$$

Under some regularity conditions, this relation can be inverted in

$$\varphi(\lambda) = \sup_\rho [\lambda\rho - I(\rho)]$$

The ***Contraction Principle***

For a mapping $x \to y$ *we have ,* $$I(y) = \inf_{x,\ x\to y} I(x)$$

that is the rate function for y is the smallest rate function (the most probable) of the values (configurations) that lead to y.

Applications

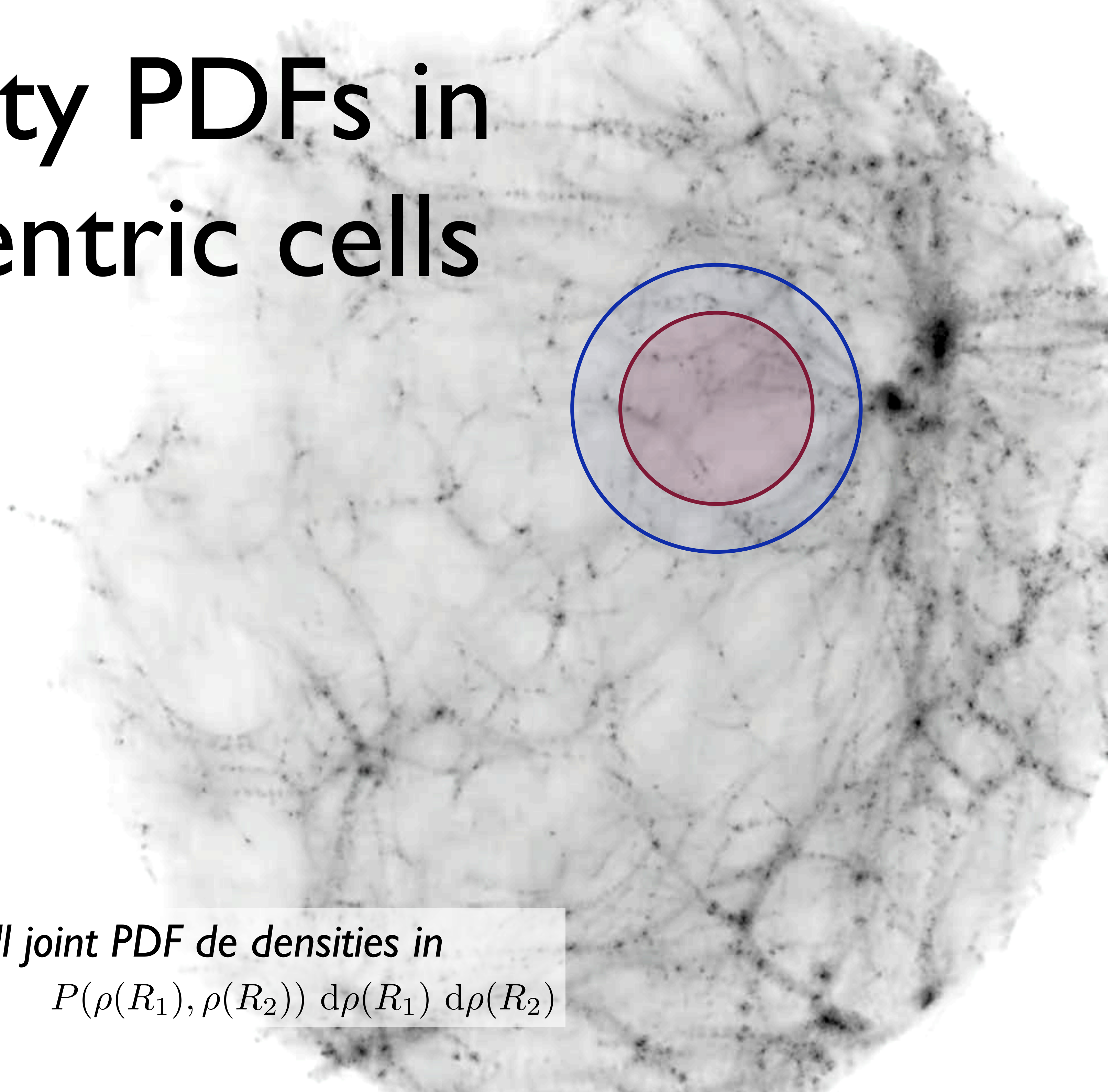
- *Shannon entropy (as rate function) and free energy (as cumulant generating function) in statistical mechanics ;*
- *Natural generalization for non-equilibrium systems (rate function for configurations) ;*
- *escape time in dynamical systems in presence of noise ;*
- *Queuing systems ;*
- *etc..*

Consequences in the context of LSS cosmology are at least 2 folds

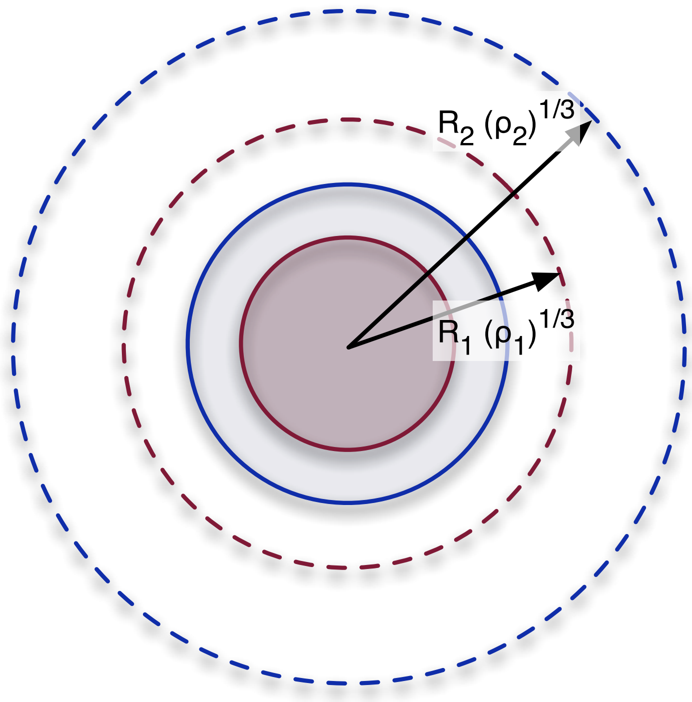
- you do not need to impose $\delta(x)$ to be small everywhere, only the variance has to be small;
- you have a possible working procedure provided you can identify the *leading* initial configuration and its probability (rate function).

In practice, such an identification can be done only for configurations with enough symmetries

Density PDFs in concentric cells



description of full joint PDF de densities in concentric cells $P(\rho(R_1), \rho(R_2))\, \mathrm{d}\rho(R_1)\, \mathrm{d}\rho(R_2)$



The rate functions (from the contraction principle)

$$\Psi(\{\rho_i\}) = \frac{1}{2}\sum_{ij}\Xi_{ij}\,\tau_i\tau_j$$

with

$$\sigma^2(R_i\rho_i^{1/3}, R_j\rho_j^{1/3})\ \Xi_{jk} = \delta_{ik}$$

The matrix Ξ is given by the inverse of correlation matrix of the density between cells at Lagrangian radius.

For spherical symmetry there exists a function ζ that gives the density ρ as a function of the linear density contrast τ

initially implemented in FB' 94, FB & Valageas '00 and developed in Valageas '02

The final expression of the scaled cumulant generating function is then given by

$$\varphi(\{\lambda_i\}) = \sum_i \lambda_i\rho_i - \Psi(\{\rho_i\})$$

with stationary conditions

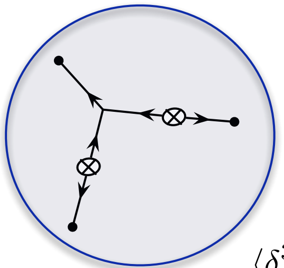
$$\lambda_i = \frac{\partial\Psi(\{\rho_i\})}{\partial\rho_i}$$

Connexion with diagrams in standard PT

scaled cumulant GF: $$\varphi(\lambda) = \lim_{\langle\rho^2\rangle_c \to 0} \langle\rho^2\rangle_c \sum_{p=1}^{\infty} \frac{\langle\rho^p\rangle_c}{p!} \left(\frac{\lambda}{\langle\rho^2\rangle_c}\right)^p = \lambda + \frac{\lambda^2}{2} + S_3 \frac{\lambda^3}{3!} + \dots$$

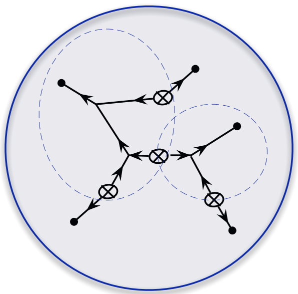
Average of (combination of) tree order expression of the p-point correlation functions in spherical cells.

Expression of $S_p = \lim_{\langle\delta^2\rangle_c \to 0} \frac{\langle\delta^p\rangle_c}{\langle\delta^2\rangle_c^{p-1}}$ *= tree order expr.*



$$\begin{aligned}\langle\delta^3\rangle &= 6\int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^3} P(k_1)P(k_2) \\ &\quad \times F_2(\mathbf{k}_1,\mathbf{k}_2) W(k_1R) W(k_2R) W(|\mathbf{k}_1+\mathbf{k}_2|R) \\ &\propto \langle\delta^2\rangle^2\end{aligned}$$

...



it has a non trivial dependence on the wave vectors through the functions F3 and F2

$$S_3 = \frac{34}{7} + \gamma_1,$$

$$S_4 = \frac{60712}{1323} + \frac{62\,\gamma_1}{3} + \frac{7\,\gamma_1^2}{3} + \frac{2\,\gamma_2}{3},$$

$$S_5 = \frac{200575880}{305613} + \frac{1847200\,\gamma_1}{3969} + \frac{6940\,\gamma_1^2}{63} + \frac{235\,\gamma_1^3}{27} + \frac{1490\,\gamma_2}{63} + \frac{50\,\gamma_1\,\gamma_2}{9} + \frac{10\,\gamma_3}{27},$$

$$\gamma_p = \frac{\mathrm{d}^p \log\sigma^2(R_0)}{\mathrm{d}\log^p R_0}.$$

1-cell density cumulants (FB '94)

Identification of initial configuration for general profiles

Considering the statistical properties of $$\rho_w = \sum_i w_i\, \rho(< R_i)$$

its scaled cumulant generating function is $$\varphi(\lambda) = \sup_{\{\tau_i\}} \left[\lambda \sum_i w_i \zeta(\tau_i) - \Psi\left(\{\tau_i\}\right)\right]$$

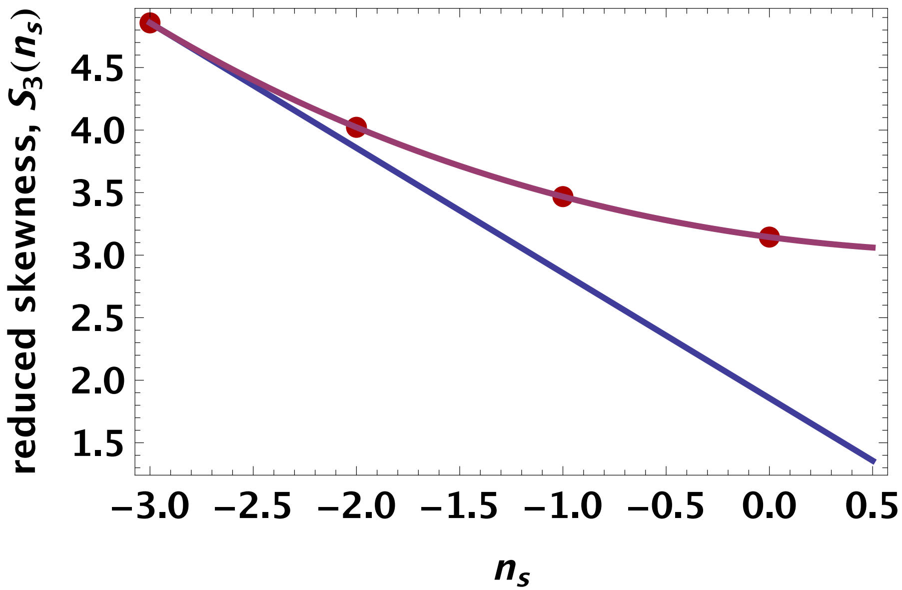
(looking for most likely configuration with Lag. mult)

Consequences

$$S_3 = 3\nu_2 \frac{\int dx\ w(x)\ \Pi^2(x)}{\left[\int dx\ w(x)\ \Pi(x)\right]^2} + \frac{\int dx\ w(x)\ x \frac{d}{dx}\Pi^2(x)}{\left[\int dx\ w(x)\ \Pi(x)\right]^2}$$

with

$$\Pi(x) = \int dy\ \Xi(x,y)\, w(y)$$



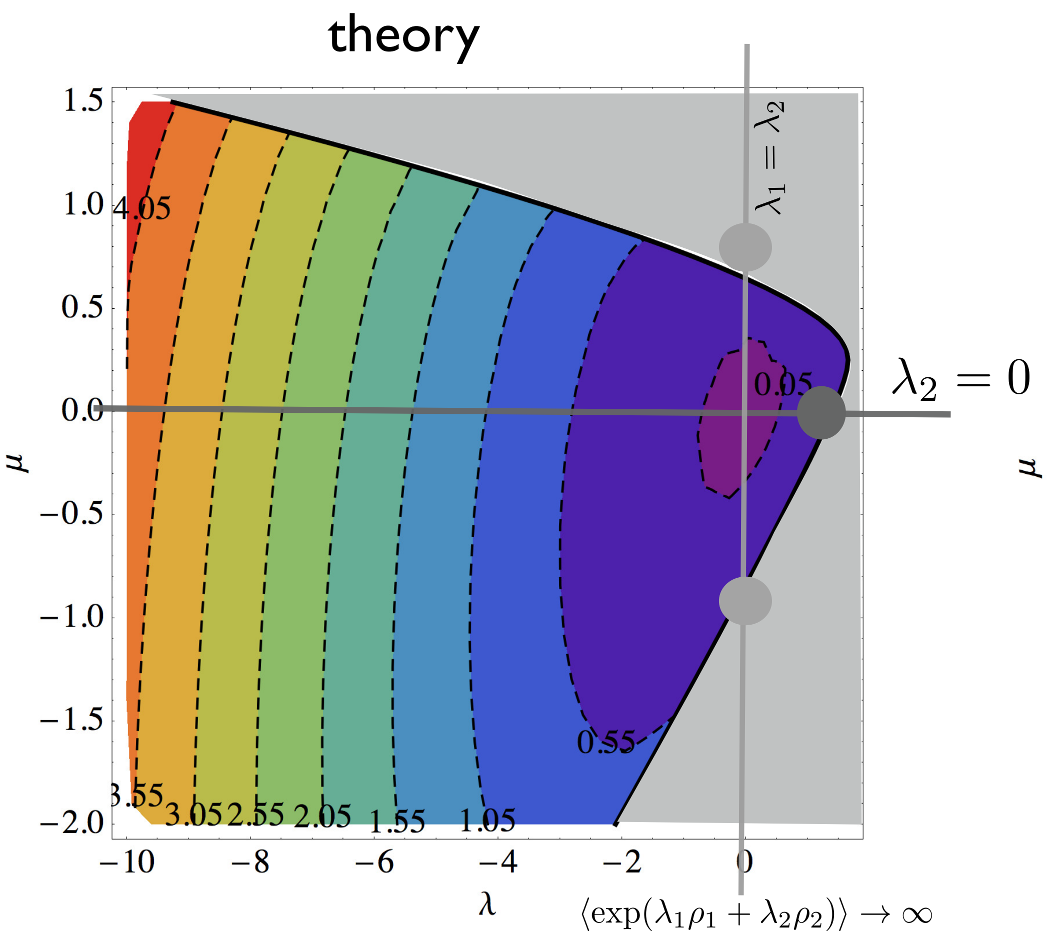
Gaussian filter, points by Juszkiewicz, Bouchet, Colombi '93 obtained from direct calculation for specific power law spectra.

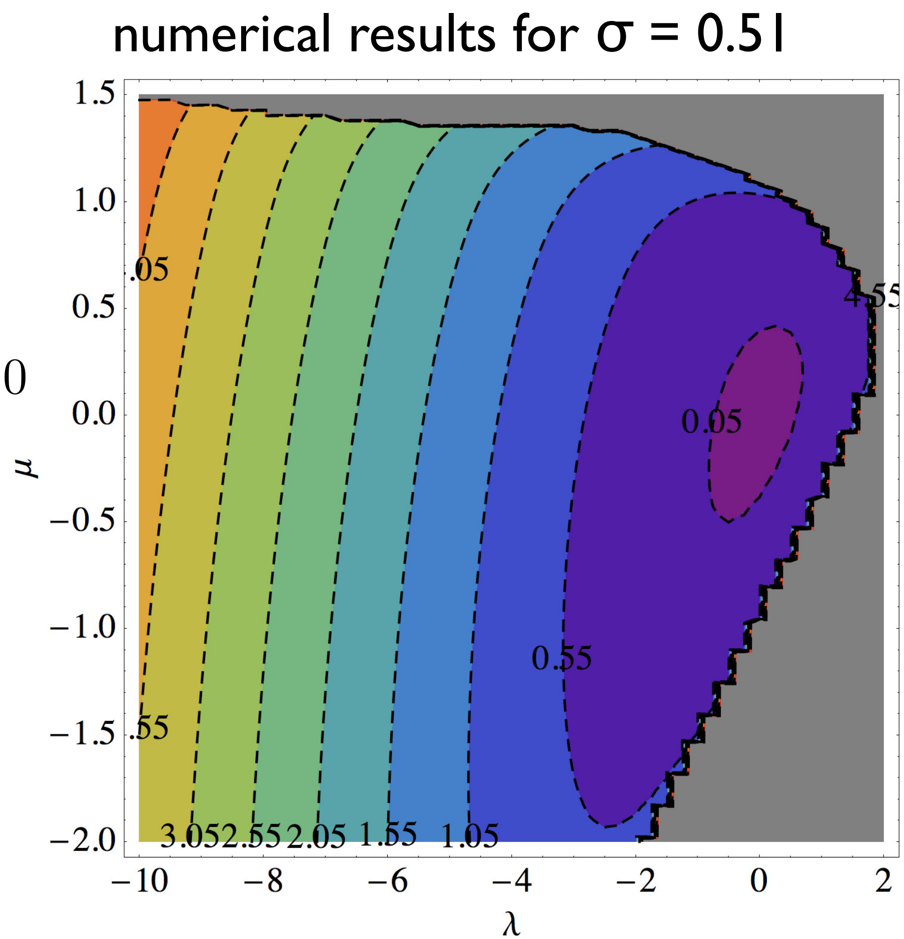
Top-hat filter, FB '94

The 2 cell cumulant generating function

The global shape of the joint cumulant generating function

FB Pichon, Codis '13

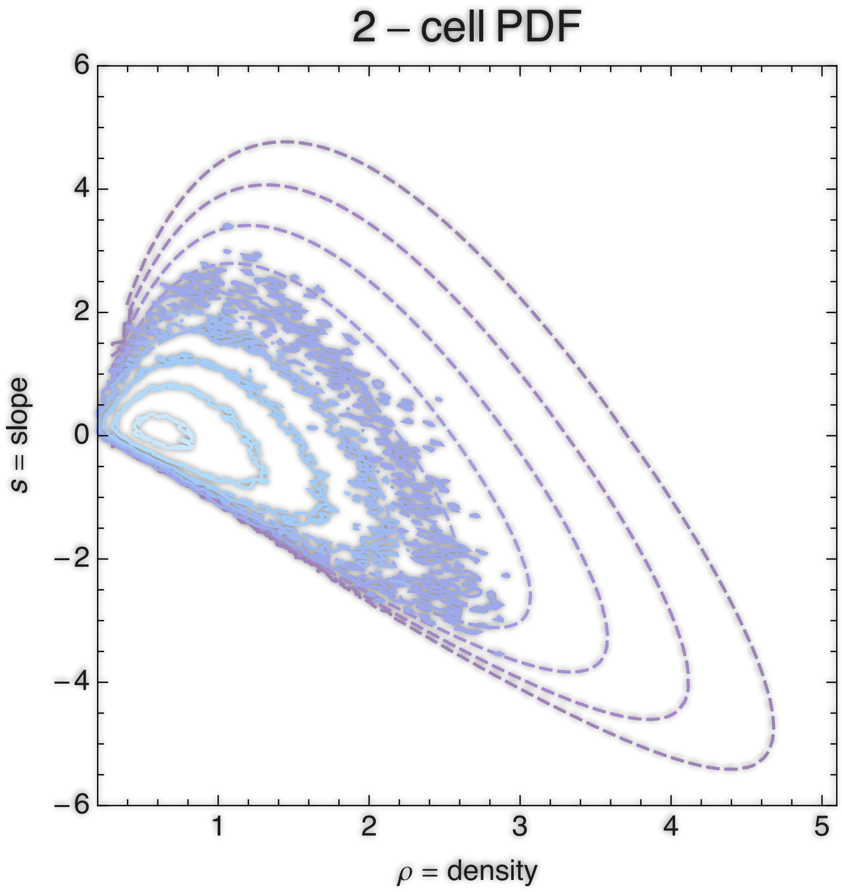


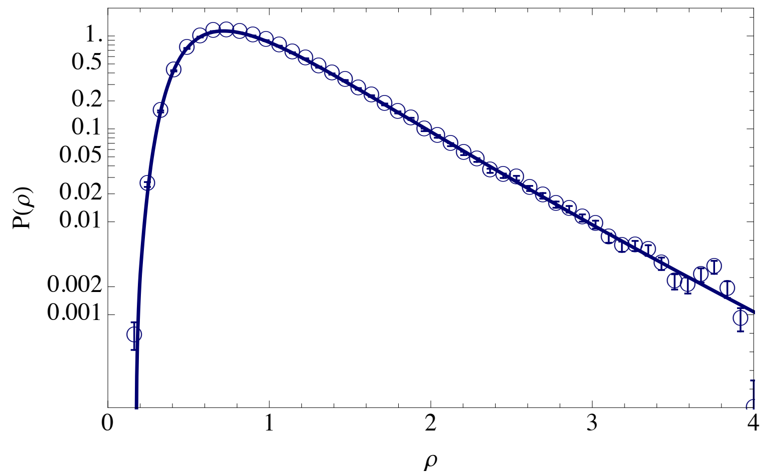


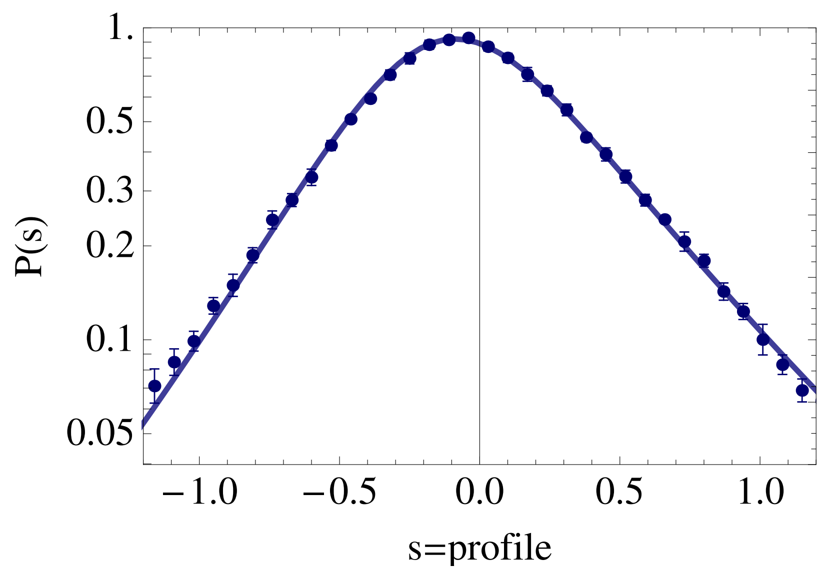
critical lines = stationary constraint is singular / signal to noise > 10%

From cumulant to PDFs

FB, Pichon, Codis '15







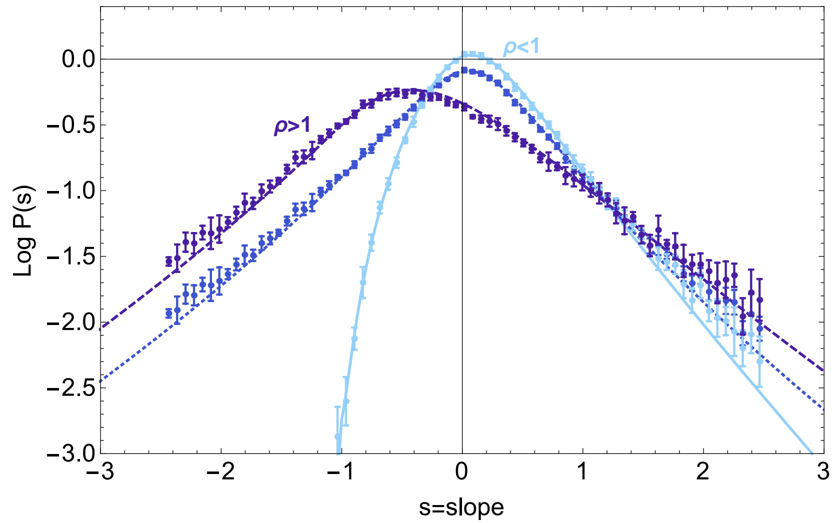


Figure 3. Density profiles in underdense (solid light blue), overdense (dashed purple) and all regions (dashed blue) for cells of radii $R_1 = 10$ Mpc/h and $R_2 = 11$ Mpc/h at redshift $z = 0.97$. Predictions are successfully compared to measurements in simulations (points with error bars).

Towards a complete theory of count-in-cell statistics…

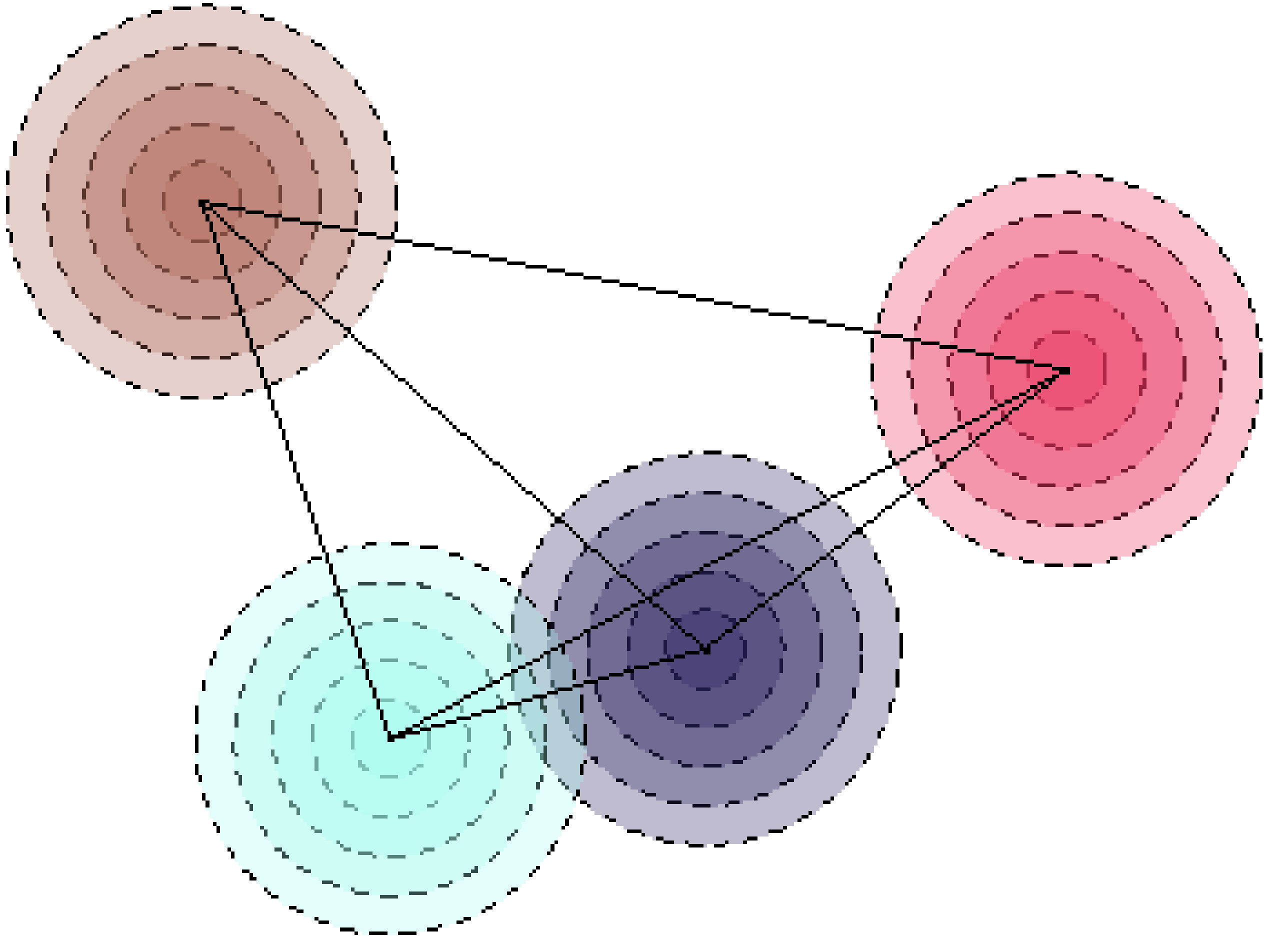


Figure 1. the configuration of multiple concentric count in cell statistics.

$$\mathcal{P}(\{\hat\rho_k\},\{\hat\rho'_k\};r_e) = \mathcal{P}(\{\hat\rho_k\})\mathcal{P}(\{\hat\rho'_k\})\left[1+\xi(r_e)b(\{\hat\rho_k\})b(\{\hat\rho'_k\})\right]$$

A regime of large-deviation functions can be identified in LSS cosmology.

- Observables can be related to joint PDFs of the density in concentric cells but also to the cumulant generating function.
- Natural application of these approaches is the density and profile PDFs

Perspectives:

- These calculations can be applied to 3D and projected mass maps, and to join density of multiple tracers;
- biasing of over-dense/under-dense regions can also be computed = statistical properties of clipped regions;

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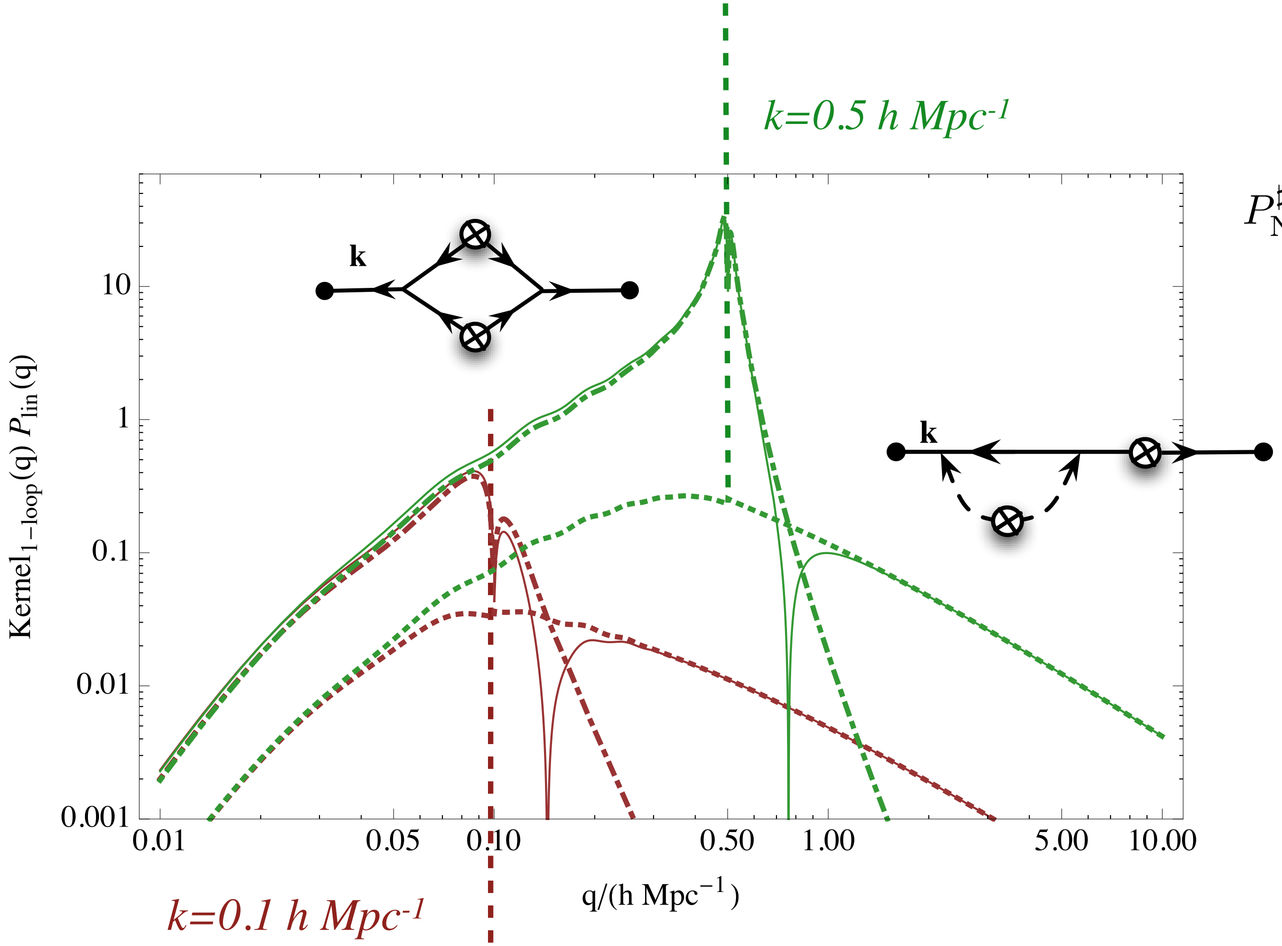
large-deviation regime

where most of the action is

Kernels in Perturbation Theory calculations

FB, Taruya, Nishimichi, '12



$$P_{\rm NL}^{\sharp-loop}(k) = \int \frac{{\rm d}q}{q}\; K^{\sharp-loop}(k,q)\; P_{\rm lin.}(q)$$

Expression of the density kernel for the propagator at 1-loop order

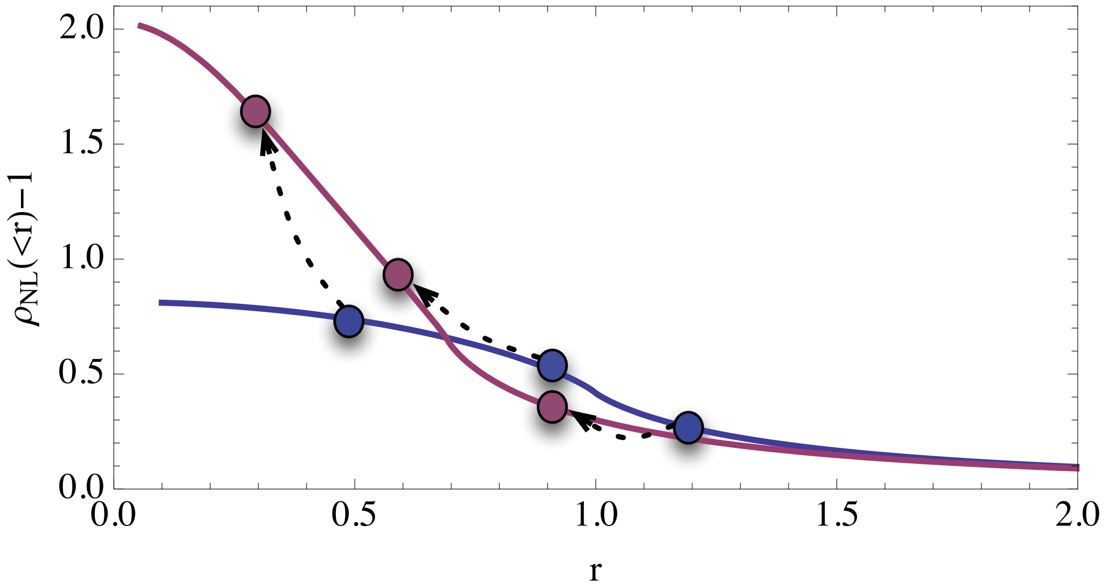
$$f(k,q) \;=\; \frac{3\left(2k^2+7q^2\right)\left(k^2-q^2\right)^3 \log\left[\frac{(k-q)^2}{(k+q)^2}\right] + 4\left(6k^7q - 79k^5q^3 + 50k^3q^5 - 21kq^7\right)}{2016k^3q^5}$$

The spherical collapse: the solution for specific initial conditions

The radius evolution

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(<R)}{R^2}$$

The exact non-linear mapping for spherically symmetric initial field (for growing mode setting)



For spherical symmetry perturbations there exists a function ζ that gives the density at time η knowing the density ρ₀ within the same Lagrangian radius at time η₀.

$$\zeta_\rho(\eta; \rho_0, \eta_0)$$

The result

The rate functions, Legendre Transform of the cumulant generating function,

$$\varphi(\{\lambda_k, R_k\}, \eta) = \sum_{p_i=0}^{\infty} \langle \Pi_i \, \hat{\rho}_{R_i}^{p_i} \rangle_c \frac{\Pi_i \lambda_i^{p_i}}{\Pi_i p_i!} \qquad \psi(\{\rho_k, R_k\}, \eta) = \sum_i \lambda_i \rho_i - \varphi(\{\lambda_k, R_k\}, \eta)$$

have, according to the contraction principle, the following time dependence,

$$\Psi(\{\rho_k, R_k\}, \eta) = \Psi\left(\{\zeta(\rho_k, \eta, \eta'), R_k \frac{\zeta^{1/3}(\rho_k, \eta, \eta')}{\rho_k^{1/3}}\}, \eta'\right)$$

In other words we know how to compute the cumulant generating function of densities in concentric cells starting with specific initial conditions.

The mathematical part, construction of the cumulant generating function

from ideas in FB' 94 see also FB & Valageas '00 and fully developed in Valageas '02

Can we get the whole generating function of the cumulants ?

$$\varphi(\{\lambda_k\}) = \sum_{p_i=0}^{\infty} \langle \Pi_i \rho_i^{p_i} \rangle_c \frac{\Pi_i \lambda_i^{p_i}}{\Pi_i p_i!}$$

It is given by the following relation (multi-dimensional Laplace transform of joint-PDFs)

$$\exp\left[\varphi(\{\lambda_i\})\right] = \langle \exp(\sum_i \lambda_i \rho_i) \rangle$$

$$= \int_0^\infty \Pi_i \mathrm{d}\,\rho_i\, P(\{\rho_i\})\ \exp(\sum_i \lambda_i \rho_i)$$

Formal solution

$$\exp\left[\varphi(\{\lambda_i\})\right] = \int \mathcal{D}\left[\tau(\vec{x})\right] \mathcal{P}\left[\tau(\vec{x})\right] \exp(\lambda_i \rho_i \left[\tau(\vec{x})\right])$$

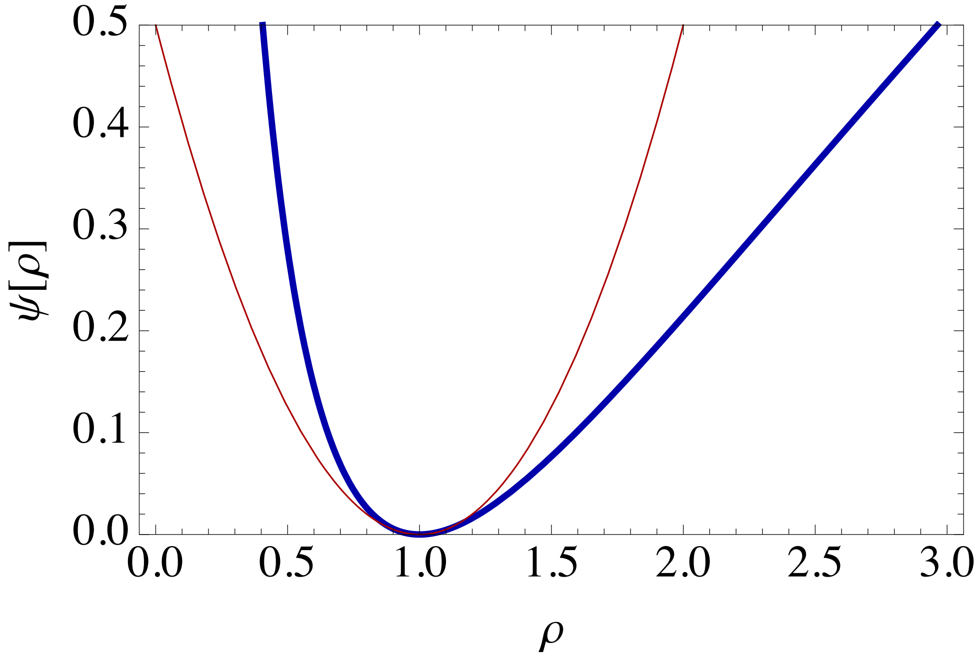
Principle of the calculations : in the small variance approximation one can look for the most probable configuration - for fixed ρ_i - and compute the resulting cumulant generating function using the steepest-descent method.

The (conjectured) solution for spherical cells: an initial spherical perturbation the profile of which can be computed from spherical collapse solution.

$$\rho_i = \zeta_{\mathrm{SC}}(\tau_i)$$

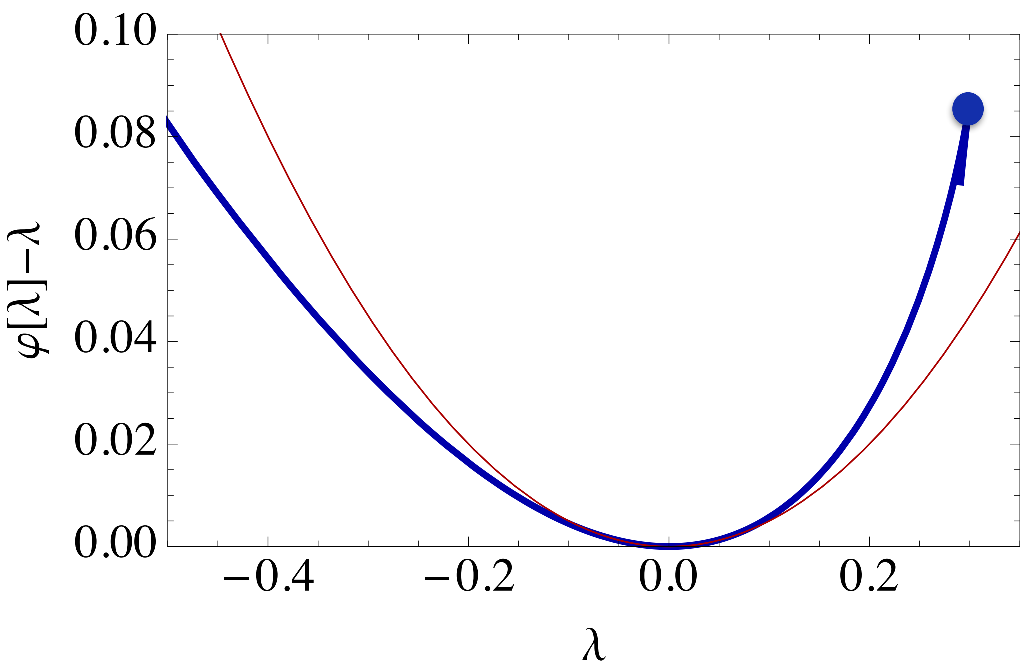
finite number of variables

The 1-cell rate function and cumulant generating function



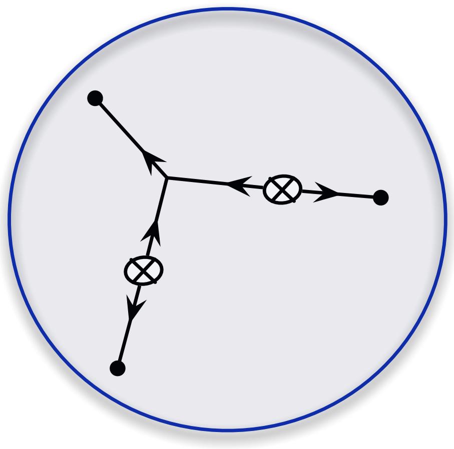
The 1-cell rate function compared to Gaussian approximation.

critical point



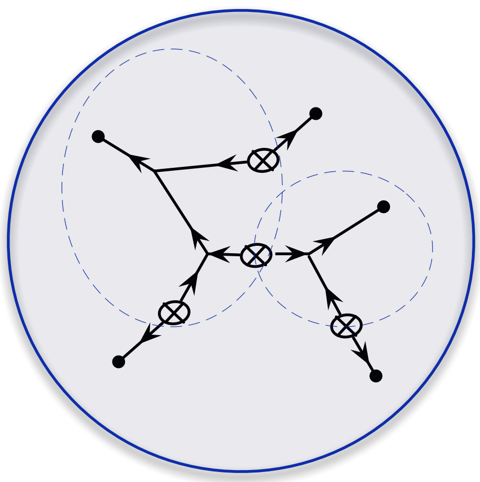
The corresponding Legendre transform.

Example of contribution to the 3- to n-point cumulants at tree order



$$\langle\delta^3\rangle = 6\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^3}P(k_1)P(k_2)$$
$$\times F_2(\mathbf{k}_1,\mathbf{k}_2)W(k_1R)W(k_2R)W(|\mathbf{k}_1+\mathbf{k}_2|R)$$
$$\propto \langle\delta^2\rangle^2$$

...



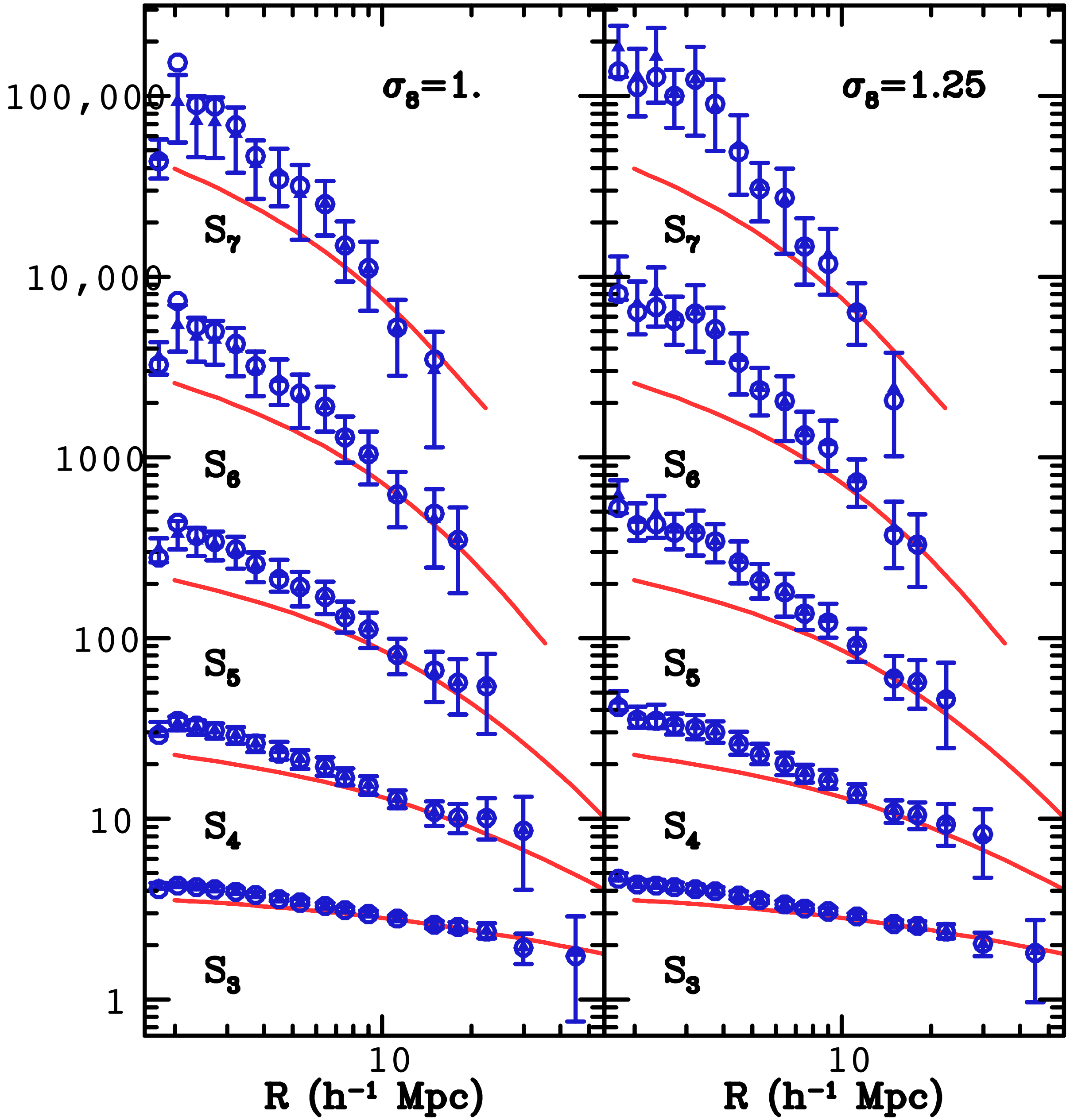
it has a non trivial dependence on the wave vectors through the functions F3 and F2

$$\langle\delta^p\rangle_c \propto \langle\delta^2\rangle_c^{p-1}$$

Predictions for cumulants and PDFs...

Baugh & Gaztañaga '95

Prediction at tree order is very accurate



Let us assume that,

$$\varphi(\lambda;\sigma) = \frac{1}{\sigma^2}\varphi_c(\sigma^2\lambda)$$

Application 1: 1-cell PDF and stats

FB Pichon, Codis '13

The inverse Laplace transform,

$$\mathcal{P}(\hat{\rho}_1) = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}\lambda_1}{2\pi\mathrm{i}} \exp(-\lambda_1 \hat{\rho}_1 + \varphi(\lambda_1))$$

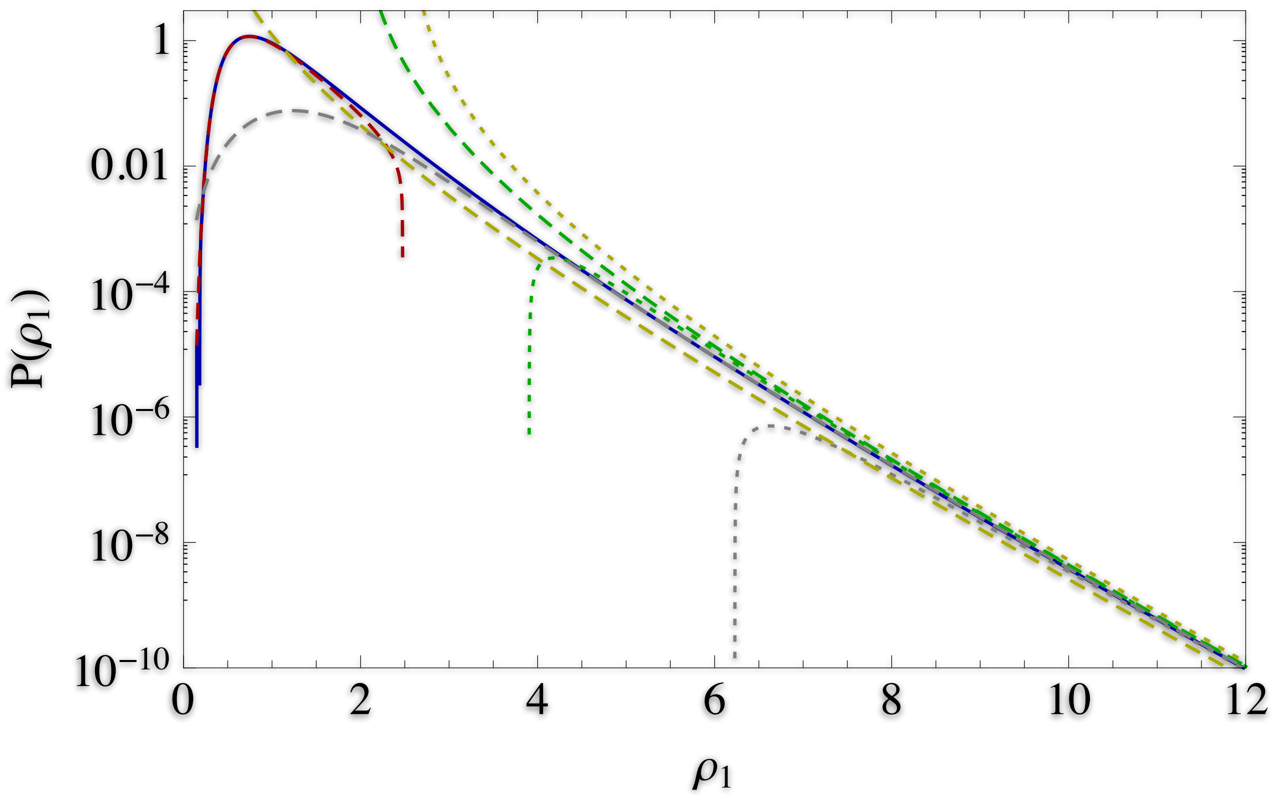
requires integration into complex plane.

$$P(\rho) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\partial^2 \Psi(\rho)}{\partial \rho^2}} \exp\left[-\Psi(\rho)\right]$$

low density approximation

$$P(\rho) = \frac{3a_{\frac{3}{2}}}{4\sqrt{\pi}} \exp\left(\varphi^{(c)} - \lambda^{(c)}\rho\right) \frac{1}{(\rho + r_1 + r_2/\rho + \ldots)^{5/2}}$$

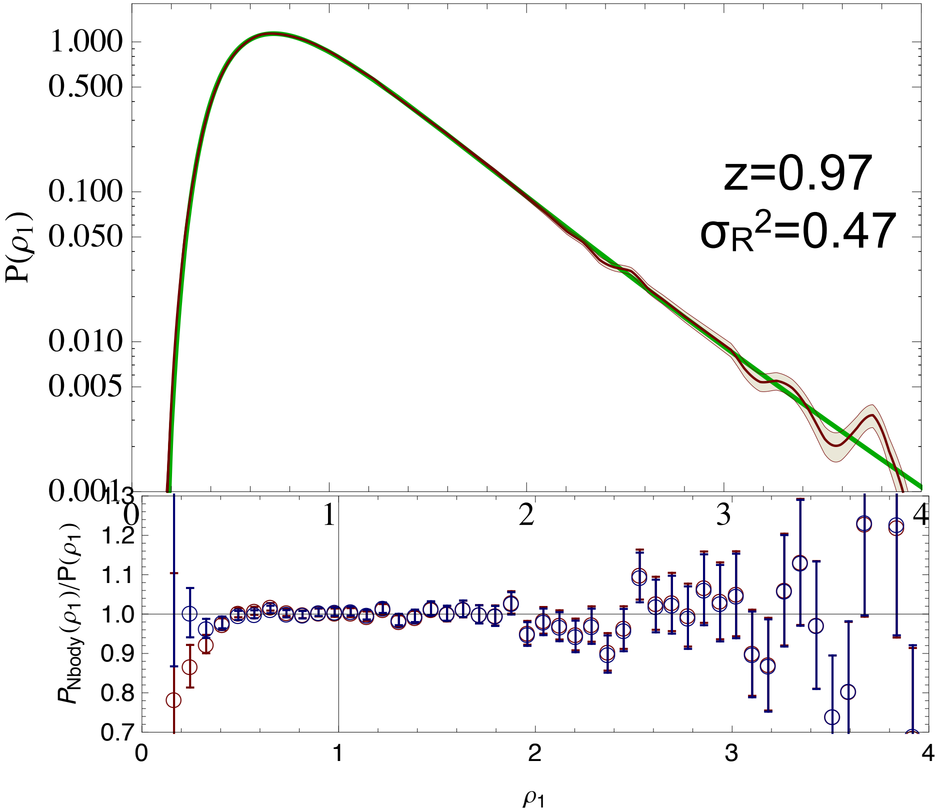
large density approximation

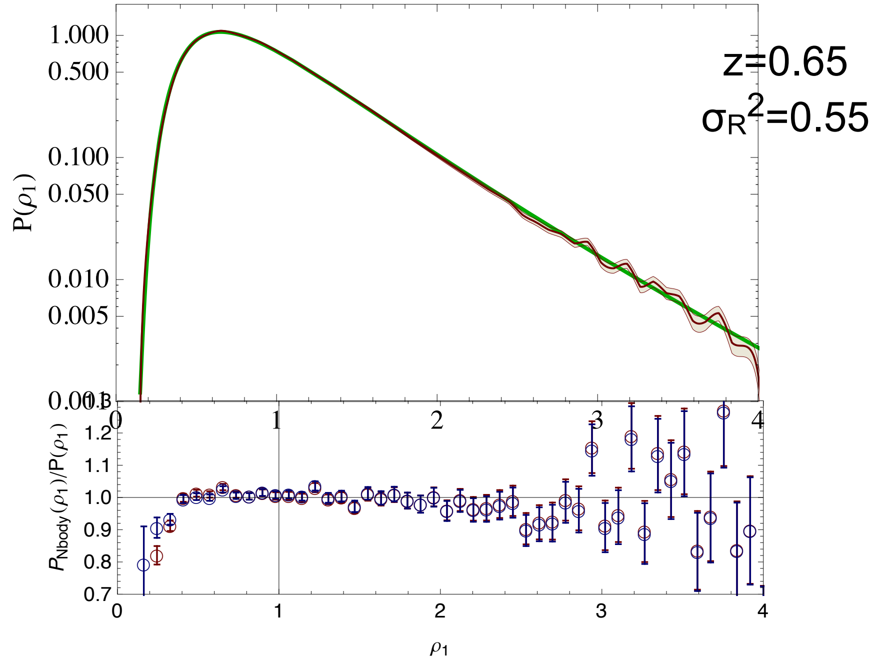


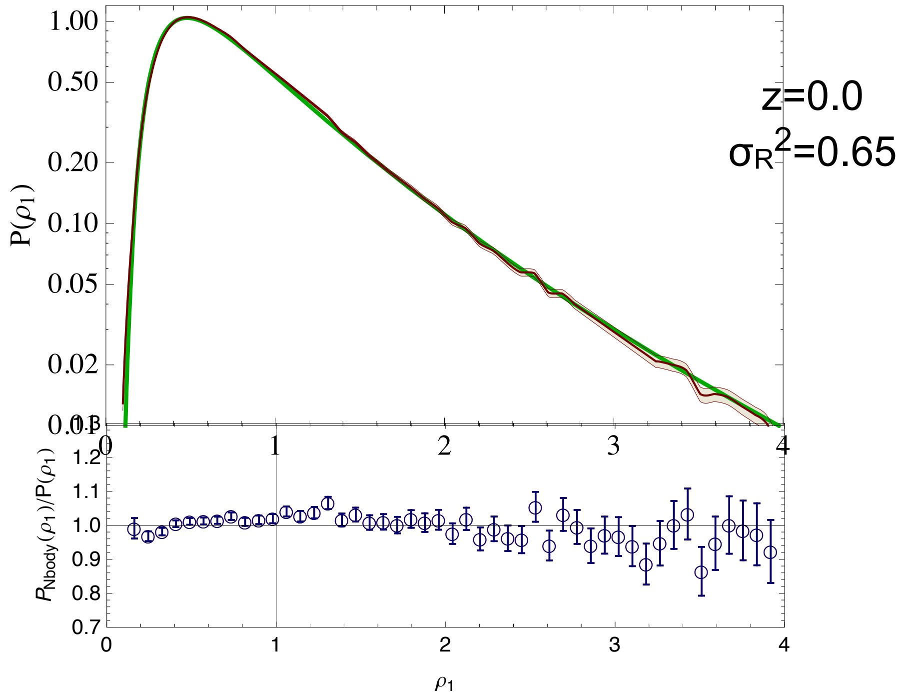
Comparison with simulations: the 1-point PDF shape

(500 h^(-1) Mpc)^3

R = 10 h^{-1} Mpc

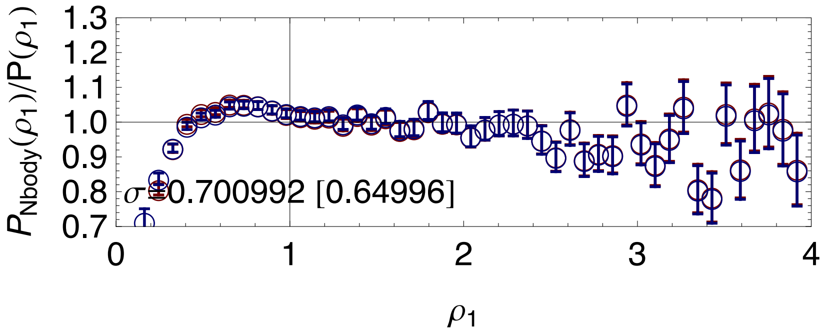


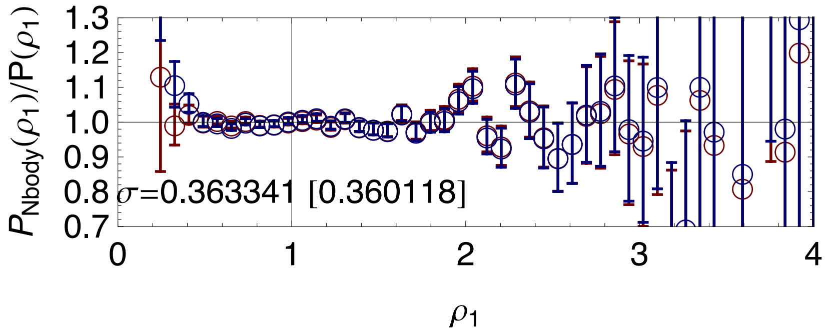


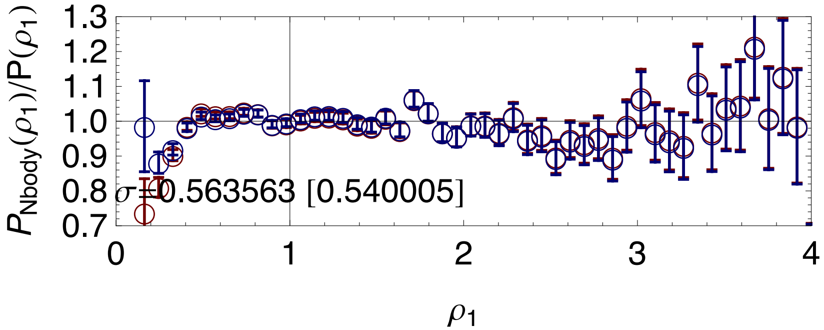


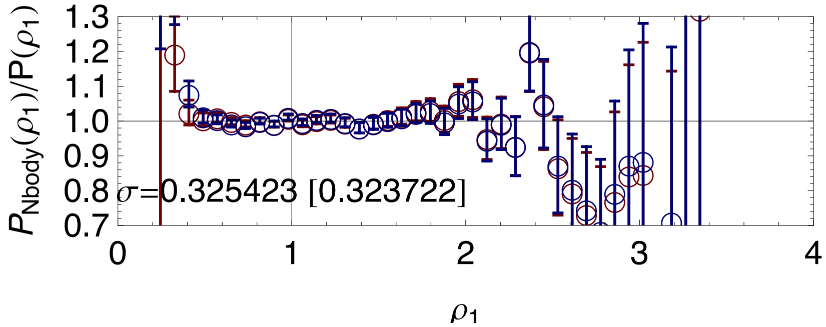
FB, Pichon, Codis '13

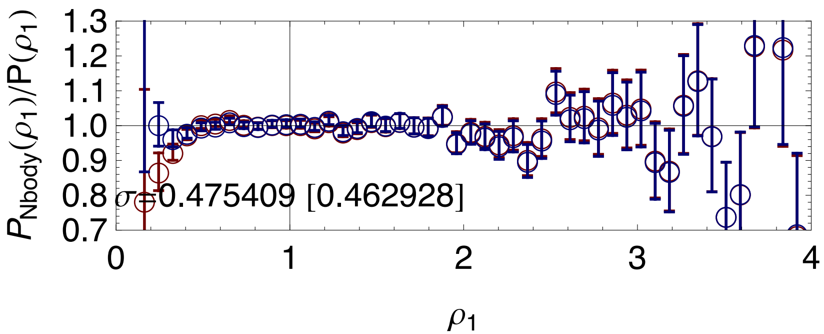
Residuals as a function of R











Effect of scale dependent index